A Splitting Approximation for the Solution of a Self-Adjoint Quenching Problem

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Outline

- System description
- Governing equations
- Literature review on recent progress on similar subject
- Our goal
- Discretization schemes
- Positivity, monotonocity and linear stability
- Simulation results
- Conclusion and Future work

System description

- Ignition process: rate of change of the temperature blows up in a very short time interval.
- Fuel combustion chamber as an example
- A non-linear advection-diffusion equation is used to model the temperature dynamics where its first derivative tends to infinity as it reaches a certain critical point.

Advection-diffusion equation

- Let $b > 0$ and $\tilde{\mathcal{D}} = (0, b) \times (0, b)$ be a squared domain, with $\partial\tilde{\mathcal{D}}$ being its boundary. We consider the following semilinear advection-diffusion problem,

$$\begin{aligned} u_t &= \nabla(a\nabla u) + f(u), \quad (\tilde{x}, \tilde{y}) \in \tilde{\mathcal{D}}, \ t > 0, \\ u(\tilde{x}, \tilde{y}, t) &= 0, \quad (\tilde{x}, \tilde{y}) \in \partial\tilde{\mathcal{D}}, \ t \geq 0, \\ u(\tilde{x}, \tilde{y}, 0) &= u_0(\tilde{x}, \tilde{y}), \quad (\tilde{x}, \tilde{y}) \in \tilde{\mathcal{D}}, \end{aligned}$$

- The nonlinear source function $f(u)$ has following properties:

$$f(0) = f_0 > 0;$$

$f(u)$ is strictly increasing for $0 \leq u < 1$;

$$\lim_{u \to 1^-} f(u) = \infty.$$

Normalized equation

- Re-scaling the space domain

$$
\begin{aligned}
u_t &= b^{-2}\,\nabla(a\nabla u) + f(u),\ (x,y)\in\mathcal{D},\ t>0, && (1.1)\\
u(x,y,t) &= 0,\quad (x,y)\in\partial\mathcal{D},\ t\geq 0, && (1.2)\\
u(x,y,0) &= u_0(x,y),\quad (x,y)\in\mathcal{D}, && (1.3)
\end{aligned}
$$

where $\mathcal{D}=(0,1)\times(0,1)$.

Semi-discretized equation

- Discrization in space only:

$$
\begin{aligned}
(v_t)_{k,\ell} = & \frac{1}{b^2 h^2}\Big[a_{k-\frac{1}{2},\ell} v_{k-1,\ell} + a_{k+\frac{1}{2},\ell} v_{k+1,\ell} \\
& - \left(a_{k-\frac{1}{2},\ell} + a_{k+\frac{1}{2},\ell}\right) v_{k,\ell} \\
& + a_{k,\ell-\frac{1}{2}} v_{k,\ell-1} + a_{k,\ell+\frac{1}{2}} v_{k,\ell+1} \\
& - \left(a_{k,\ell-\frac{1}{2}} + a_{k,\ell+\frac{1}{2}}\right) v_{k,\ell}\Big] \\
& + f(v_{k,\ell})
\end{aligned}
$$

for $1 \leq k, \ell \leq N$.

Semi-discrete equation matrix form

- Initial value problem

$$
\begin{aligned}
v' &= (P+Q)v + g(v), \quad t>0, && (1.4)\\
v(0) &= v_0, && (1.5)
\end{aligned}
$$

in which

$$
P = \frac{1}{b^2h^2}\text{diag}\left(P_1, P_2, \ldots, P_N\right)
$$

$$
Q = \frac{1}{b^2h^2}\text{tridiag}\left(Q_j^1, Q_j^2, Q_j^3\right)
$$

Exact solution of the semi-discrete equation

- The exact solution of the initial value problem (1.4), (1.5) is

$$v(t) = e^{t(P+Q)}v_0 + \int_0^t e^{(t-\tau)(P+Q)}g(v)d\tau,$$

  $0 \le t \le T < \infty.$

- The Peaceman-Rachford splitting is used to approximate the exponential, i.e
  $e^{t(P+Q)} = S(t) + \mathcal{O}(t^3)$ where

$$S(t) = (I - \frac{t}{2}Q)^{-1}(I - \frac{t}{2}P)^{-1}(I + \frac{t}{2}P)(I + \frac{t}{2}Q),$$

  and the trapezoidal rule for approximating the integral. The following scheme is then obtained where τ is a fixed time step:

Discretized equation

- Full discretized equation

$$v^{n+1} = \tag{1.6}$$

$$(I - \frac{\tau}{2}Q)^{-1}(I - \frac{\tau}{2}P)^{-1}(I + \frac{\tau}{2}P)(I + \frac{\tau}{2}Q)\left(v^n + \frac{\tau}{2}g(v^n)\right) \tag{1.7}$$

$$+\frac{\tau}{2}g(w^{(n)}) + \mathcal{O}(\tau^2). \tag{1.8}$$

where an Euler scheme is used for the approximation $v^{n+1} \approx w^{(n)} = v^n + \tau((Q+P)v^n + g(v^n))$

- Lemma 1. $\max(\|P\|, \|Q\|) \leq \dfrac{4}{b^2h^2}\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}$.

  Proof:

  We have:

  $$\|P\| = \frac{1}{b^2h^2}\max_j\{\max\{u_{1,j} - m_{1,j}, l_{N-1,j} - m_{N,j}, \max_{i=2,\dots N-1}(l_{i,j} - m_{i,j} + u_{i,j+1})\}\} \leq \frac{4}{b^2h^2}\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}.$$

  Similarly the same bound can be obtained for $||Q||$.

Monotonicity and positivity: lemma 2

- Lemma 2. If $\frac{\tau}{h^2} < \frac{b^2}{2\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}$ then the matrices $(I - \frac{\tau}{2}Q), (I - \frac{\tau}{2}P), (I + \frac{\tau}{2}P)$ and $(I + \frac{\tau}{2}Q)$ are non-singular. Also $(I - \frac{\tau}{2}R)$ and $(I - \frac{\tau}{2}P)$ are monotone and inverse positive. $(I + \frac{\tau}{2}Q)$ and $(I + \frac{\tau}{2}P)$ are nonnegative. Similarly, If $\frac{\tau}{h^2} < \frac{b^2}{8\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}$, then $I + \tau(P + Q)$ is nonsingular as well as nonnegative.

Proof.
According to lemma 1
$\|\frac{\tau}{2}P\| \leq \frac{2\tau}{b^2 h^2}\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\} < 1.$
Hence $(I + \frac{\tau}{2}P)$ is nonsingular [2]. It can be observed that $(I + \frac{\tau}{2}P)$ is also nonnegative. A similar argument shows that $(I + \frac{\tau}{2}Q)$ is nonsingular and nonnegative.
Again according to lemma (10) we have
$\|\tau(P+Q)\| \leq \tau\left(\|P\| + \|Q\|\right) \leq \frac{8\tau}{h^2 b^2}\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\} < 1$
Therefore $I + \tau\,(P+R)$ is nonsingular and nonnegative.
Consider the matrix $M = I - \frac{\tau}{2}P$. As $M_{i,j} \leq 0$ for $i \neq j$ and the weak row sum criterion is satisfied M is monotone and hence an inverse exists which is nonnegative [2]. Using a similar argument shows that $I - \frac{\tau}{2}Q$ is inverse-positive. □

Monotonicity and positivity: lemma 3-4

- Lemma3 Let $(P+Q)v^l + g(v^l) + \frac{\tau^2}{4} PQg(v^l) > 0$, where $l \geq 0$ is any begining time step. For uniform time steps τ , if $\frac{\tau}{h^2} < \frac{b^2}{2\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}$ then $v^{k+1} > v^k$ for all $k \geq l$. The sequence $\{v^k\}_{k=l}^{\infty}$ is therefore monotonically increasing.

- lemma4 If $v_0 = 0$, and $\frac{\tau}{h^2} < \frac{b^2}{\max_{i,j}\left\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\right\}}$, then $(P+Q)v_0 + g(v_0) + \frac{\tau_0^2}{4} PQg(v_0) > 0.$

Monotonicity and positivity: lemma 5 and theorem

- Lemma5 If $\frac{\tau_0}{h^2} < \frac{b^2}{2\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}$

  $h^2 < \frac{1}{4M\frac{b^2}{2\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}}$

  then given $\mathbf{v_0} = 0$ all the components of $\mathbf{v^1} < 1$.
  where $M = f(\tau_0 f_0)$.
- Theorem

  For any beginning step τ, if:

  (i) $\frac{\tau}{h^2} < \frac{b^2}{2\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}$

  (ii)$(P+R)v_0 + g(v_0) + \frac{\tau^2}{4}PRg(v_0) > 0$

  then the sequence of solutions $(v^k)_{k\geq l}$ produced by the Peaceman-Rachford-Strang splitting increases monotonically until unity is exceed by at least component of the solution vector(i.e until quenching occurs).

Linear stability

Stability is a challenge while solving non-linear blow-up type or quenching-type problem such as the one treated in this work. Analysis by mean of linear stability is not a vigorous process, but it is still a useful and effective practice. This linear stability method freezes the non linear term. In other words the non linear term is kept constant.

Linear stability: lemma 6

- Lemma 6 $\|(I - \frac{\tau}{2}P)^{-1}\|, \|(I - \frac{\tau}{2}Q)^{-1}\| \leq 1$

  proof: Consider the entries of the matrix $(I - \frac{\tau_k}{2}P)$:

  $\alpha_j^{(i)} = 1 - \frac{\tau}{2}(m_j + u_j + l_{j-1})$

  Since the matrix P is diagonal dominant with diagonal entries $m_j < 0$,then $\alpha_j^{(i)} \geq 1$.

  The desired result is obtained by using the Varah-bound [1].

  Similarly,

  $\beta_j^{(i)} = 1 - \frac{\tau}{2}(m_i + u_i + l_{i-1}) \geq 1$, and as a consequence $\|(I - \frac{\tau}{2}R)^{-1}\| \leq 1$.

- Lemma 7. If $\frac{\tau}{h^2} < \frac{b^2}{2\max_{i,j}\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\}}$
  Then $\|(I + \frac{\tau}{2}P)(I + \frac{\tau}{2}R)\| = 1$
  Proof
  Since the matrices $(I + \frac{\tau}{2}P)$, and $(I + \frac{\tau}{2}R)$ are non-negative, so is the product $(I + \frac{\tau}{2}P)(I + \frac{\tau}{2}R)$. For any non-negative vector such that $\mathbf{x} \leq 1$, we have
  $0 \leq (I + \frac{\tau}{2}(P + R) + \frac{\tau^2}{4}PR)\mathbf{x} \leq 1.$
  Since $\frac{\tau}{2}(P + R) + \frac{\tau^2}{4}PR)\mathbf{x}$ has some zero entries, we can deduce that $\|(I + \frac{\tau}{2}P)(I + \frac{\tau}{2}R)\|_\infty = 1.$

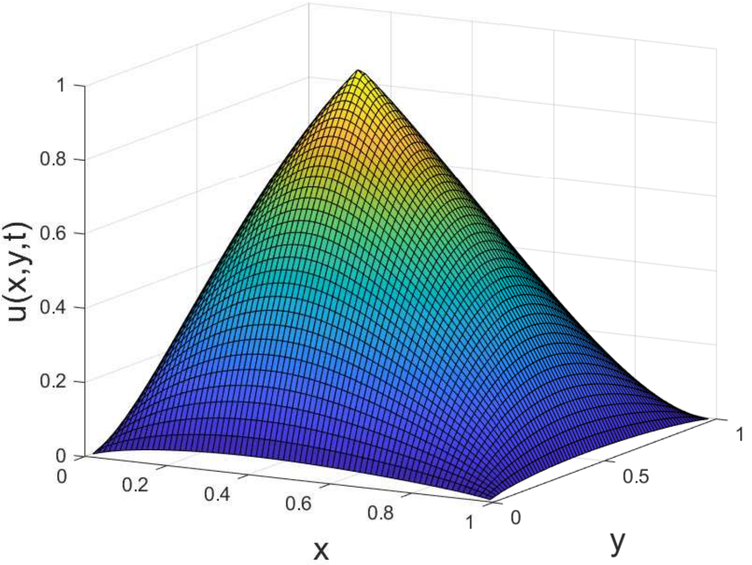
Linear stability: lemma 8

- Lemma 8. Let $\frac{\tau}{h^2} < \frac{b^2}{2\max_{i,j}\left\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\right\}}$. Then the Peaceman-Rachford spliting is weakly stable in the Von Neumann sense.
  Proof,
  Consider a perturbation at time step k , $z_{k+1} = v_k - \tilde{v}_k$, where $\tilde{v}_k$ is the computed solution. By lemmas (16)-(17) the norm is bounded.
  $\|\mathbf{z_{k+1}}\| = \|(I - \frac{\tau}{2}R)^{-1}(I - \frac{\tau}{2}P)^{-1}(I + \frac{\tau}{2}P)(I + \frac{\tau}{2}R)\mathbf{z_k}\|$
  $\leq \|(I - \frac{\tau}{2}R)^{-1}(I - \frac{\tau}{2}P)^{-1}(I + \frac{\tau}{2}P)(I + \frac{\tau}{2}R)\|\|\mathbf{z_k}\| \leq \|\mathbf{z_k}\|.$

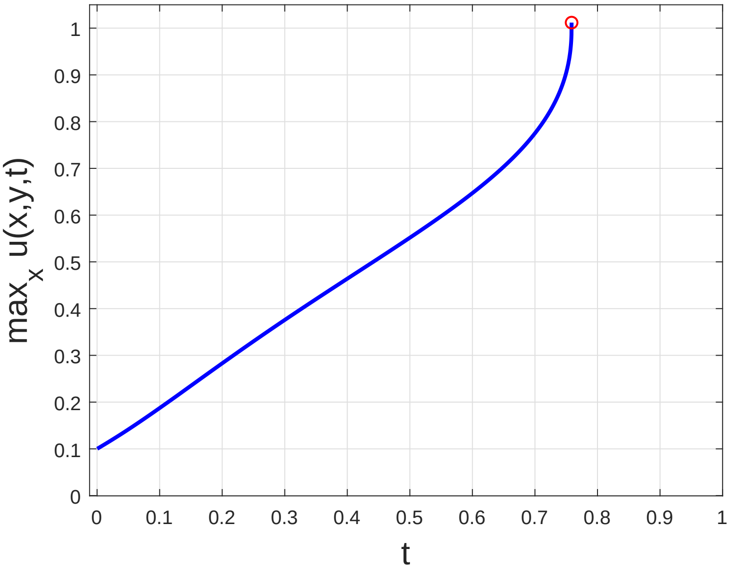
Linear stability: theorem

- Theorem: Let $\dfrac{\tau}{h^2} < \dfrac{b^2}{2\max_{i,j}\left\{a_{i\pm\frac{1}{2},j}, a_{i,j\pm\frac{1}{2}}\right\}}$ for all $i \le k$.
  Then the linearized variable step Peace-Rachford method is weakly stable in the von neumann sense under the l_∞ norm, i.e $\|\mathbf{z}_{k+1}\| \le \|\mathbf{z}_0\|$.
  where $\mathbf{z}_0 = \mathbf{v}_0 - \bar{\mathbf{v}}_0$ is an initial perturbation arising from the initial perturbation $\mathbf{z}_0$.

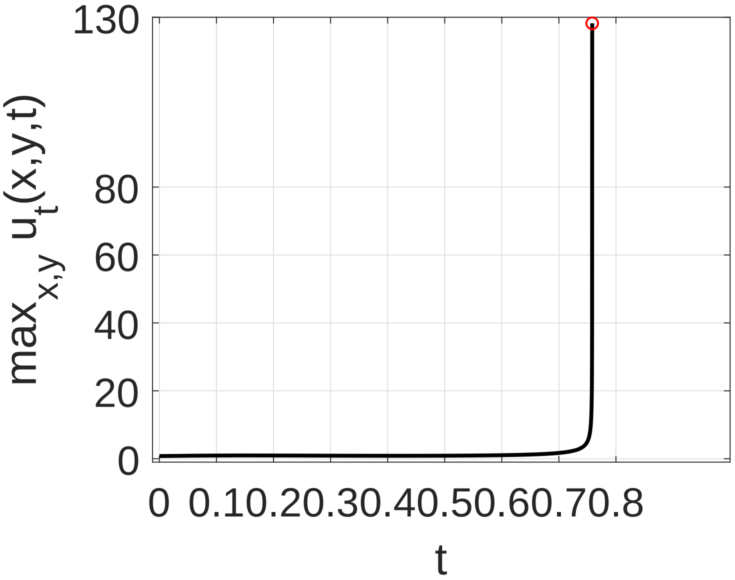
Numerical solution immediately before quenching time



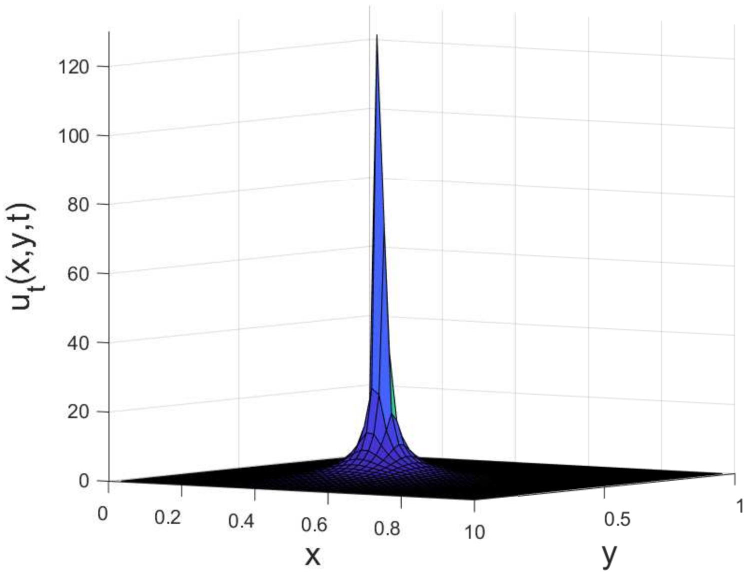
Maximum temperature values



Maximum temperature derivative values



Temperature derivative values



Summary and future work

- We have derived conditions for the physical properties of the quenching equation with variablwe difusion coeficient: positivity, monotonocity, and also linear stability.
- Using uniform grid and constant time step we showed some numerical results.
- One of our future goal is obtaining more accurate numerical results by using non-uniform grid and variable time step to capture the quenching location and time.
- Nonlinear stability is another important future goal for us to achieve

Thank you for your attention!

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