Perspectives on some problems in Analysis

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Idea of perturbation theory

Question: Given operator A on a Hilbert space $\mathcal{H}$, what can be said about the spectral properties of

$$T = A + B \quad \text{for} \quad B \in \text{Class } X?$$

- Classically Class $X = \{\text{trace cl.}\}, \{\text{Hilb.–Schmidt}\}, \{\text{comp.}\}$.
- In this talk A and B are self-adjoint and

$$\text{Class } X = \{\text{rank } M\}, \quad \text{or} \quad \{\text{random non-compact}\}.$$

- Operator $T = A_\alpha = A + B$ is a rank M perturbation of A, iff for some coordinate operator $\mathbf{B} : \mathbb{C}^M \to \mathcal{H}$ and symmetric $\alpha \in \mathbb{C}^{M\times M}$ we have $B = \mathbf{B}\alpha\mathbf{B}^*$.
- Note: If α is diagonal, then

$$B = \sum_{m=1}^{M} \alpha_m(\,\cdot\,, \varphi_m)\varphi_m$$

for some $\varphi_m \in \mathcal{H}$ and scalars $\alpha_m \in \mathbb{R}$.

Trivial examples

- $A_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \alpha(\,\cdot\,, e_1)e_1 = \begin{pmatrix} 1+\alpha & 0 \\ 0 & 3 \end{pmatrix}$ acting on $\mathbb{R}^2$.
- $A_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \alpha(\,\cdot\,, e_1+e_2)(e_1+e_2) = \begin{pmatrix} 1+\alpha & \alpha \\ \alpha & 3+\alpha \end{pmatrix}$.
- $A_{\alpha,\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + \alpha(\,\cdot\,, e_1)e_1 + \beta(\,\cdot\,, e_2)e_2 = \begin{pmatrix} 1+\alpha & 0 \\ 0 & 3+\beta \end{pmatrix}$.

For a $k \times k$ matrix, the k eigenvalues depend on the parameter(s).
Finding EVA and EVE consists of diagonalization $UA_\alpha = DU$.
Operators on infinite dimensional space (e.g. Hilbert space) reveal more complicated spectral behavior.

Spectral decompositions

Theorem (Spectral Theorem)

Let A be a self-adjoint operator on Hilbert space with cyclic vector φ. Then there exists a unique measure $\mu = \mu^{\varphi}$ such that

$$\left((A-\lambda \mathbf{I})^{-1}\varphi, \varphi\right)_{\mathcal{H}} = \left((M_t - \lambda \mathbf{I})^{-1}\mathbf{1}, \mathbf{1}\right)_{L^2(\mu)} = \int_{\mathbb{R}} \frac{d\mu(t)}{t-\lambda}$$

for $\lambda \in \mathbb{C}\backslash\mathbb{R}$. That is, there is a unitary operator $U : \mathcal{H} \to L^2(\mu)$ such that $UA = M_t U$ and $U\varphi = \mathbf{1}$. Notation: $A \sim M_t$.

- An EVE of A with EVA λ is mapped to $\chi_{\{\lambda\}}$, and $\mu(\{\lambda\}) > 0$.
- Notation $T \sim A$(Mod compact operators) means there is a unitary U and a compact B so that $UT = AU + B$.
- Lebesgue decomposition of spectral measure $d\mu = d\mu_{\mathrm{ac}} + d\mu_{\mathrm{s}}$.
- Alternatively decompose $d\mu = d\mu_{\mathrm{ess}} + d\mu_{\mathrm{d}}$.
- Analogously decompose operators $A = A_{\mathrm{ac}} \oplus A_{\mathrm{s}} = A_{\mathrm{ess}} \oplus A_{\mathrm{d}}$. and spectrum $\sigma = \sigma_{\mathrm{ac}} \dot{\cup} \sigma_{\mathrm{s}} = \sigma_{\mathrm{ess}} \dot{\cup} \sigma_{\mathrm{d}}$ with $\sigma(A) = \operatorname{supp} d\mu$.

Classical perturbation theory (A, T self-adjoint)

Theorem (Weyl–vonNeuman early 1900's)

$T \sim A$(*Mod compact operators*) $\quad\Leftrightarrow\quad \sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(T)$.

Theorem (Kato–Rosenblum 1950's, Carey–Pincus 1976)

$T \sim A$(*Mod trace class*) $\quad\Leftrightarrow\quad A_{\text{ac}} \sim T_{\text{ac}}$, *conditions.*

Theorem (Aronszajn–Donoghue Theory 1970-80's)

Spectral type is not stable under rank one perturbations. Complete info about the eigenvalues, but only a set outside which A_α *has no singular continuous spectrum. Singular spectrum moves:* $\mu_{\text{s}} \perp \mu_\alpha$.

Theorem (Poltoratski 2000)

Conditions on purely singular operators $\Rightarrow A_\alpha \sim A$(*Mod rank 1*).

Barry Simon: "The cynic might feel that I have finally sunk to my proper level [...] to rank one perturbations–maybe something so easy that I can say something useful! We'll see even this is hard and exceedingly rich."

Cyclicity

- Example 1: For $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ we have $A \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$ so

$$\operatorname{span}\left\{ \begin{pmatrix} a \\ b \end{pmatrix}, A \begin{pmatrix} a \\ b \end{pmatrix}, A^2 \begin{pmatrix} a \\ b \end{pmatrix}, \dots \right\} = \operatorname{span}\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \right\} \neq \mathbb{R}^2.$$

- Example 2: For $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ we have $A \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ 3b \end{pmatrix}$ so that

$$\operatorname{span}\left\{ \begin{pmatrix} a \\ b \end{pmatrix}, A \begin{pmatrix} a \\ b \end{pmatrix}, A^2 \begin{pmatrix} a \\ b \end{pmatrix}, \dots \right\} = \mathbb{R}^2.$$

- Cyclic subspace $[\varphi]_A := \overline{\operatorname{span}\{A^n \varphi : n \in \mathbb{N}_0\}}$.
- φ is cyclic for self-adjoint operator A on $\mathcal{H}$, if $[\varphi]_A = \mathcal{H}$.
- Operator A is cyclic, if there exists a cyclic vector.

Explore: Invariant Subspace Problem 1

Question around Beurling, von Neumann mid-1900's: Does every bounded linear operator on a complex Banach space have a NTIS?
Subspace $W \leq H$ is invariant for $T \quad :\Longleftrightarrow \quad T(W) \subset W$.

- Any eigenvector spans a NTIS.
- If $[\varphi]_T$ is non-trivial, then $[\varphi]_T$ is a NTIS of T.
- Enflo (1976/Acta 1987) presented a very complicated construction of an operator on Banach space without NTIS.
- Śliwa (Canad. Math. Bull. 2008): Any ∞-dimensional Banach space of countable type over a non-Archimedean field admits a bounded linear operator without NTIS.
- Argyros–Haydon (Acta 2011) constructed ∞-dim. Banach space on which every continuous operator is the sum of a compact and a scalar (so every operator has invariant subsp.).
- For separable Hilbert spaces the problem is still open.
- Spectral Theorem provides NTISs of self-adjoint operators.
- Operators such as the shift operator, $Sf(z) = zf(z)$, are studied on analytic function spaces.

Explore: Invariant Subspace Problem 2

Question around Beurling, von Neumann mid-1900's: Does every bounded linear operator on a complex Banach space have a NTIS?

- William DeMeo wrote: "There's an amusing comment in Peter Lax's Functional Analysis book. After a brief description of the Invariant Subspace Problem, he says (paraphrasing) '[. . .] this question is still open. It is also an open question whether or not this question is interesting.' [. . .]"
- Bill Johnson replied: "[. . .] Why is the twin prime conjecture interesting?"

Cyclic vectors for rank one perturbations

- Self-adjoint operator A on Hilbert space $\mathcal{H}$, some vector φ.
- Recall family of rank one perturbations given by

$$A_\alpha = A + \alpha(\,\cdot\,, \varphi)\varphi \text{ for } \alpha \in \mathbb{R}.$$

- WLOG: φ cyclic for A.

Theorem (Abakumov–L–Poltoratski, JLMS 2013)

Let $A_\alpha = A + \alpha(\cdot, \varphi)\varphi$ on a Hilbert space $\mathcal{H}$. If $0 \neq f \in \mathcal{H}$, then

1) *The function f is a cyclic vector for $(A_\alpha)_{\mathrm{ac}}$ for all but a countable number of $\alpha \in \mathbb{R}$.*
2) *The function f is a cyclic vector for $(A_\alpha)_{\mathrm{s}}$ for Lebesgue a.e. $\alpha \in \mathbb{R}$.*

We have an example: Function f non-cyclic for $(A_\alpha)_{\mathrm{s}}$ for uncountably many α.

Rank one perturbations in math physics

- ODEs and PDEs with changing boundary condition.
- Simon–Wolf criterion, 'singular cont. spectrum is generic', applied to classes of random Schrödinger operators shows that the parameter set with purely singular continuous spectrum is dense. No info about the probability.
- Anderson-type Hamiltonian $A_\omega = A + \sum_{m\in\mathbb{N}} \omega_m(\,\cdot\,,\varphi_m)\varphi_m$ for orthonormal φ_m and i.i.d. random ω_m wrt P. Examples:
  - Discrete random Schrödinger operator on graph,
  - free Jacobi matrix + random entries on diagonal,
  - many continuous operators of interest to math physicists.

Theorem (L, Banach J. Math. 2019)

Let μ_ω denote the spectral measure of $(A_\omega)_{\text{ess}}$ with respect to some cyclic φ_m. If $|\partial$ess-supp $(\mu_\omega)_{\text{ac}}| = 0$ almost surely, then

$$(A_\omega)_{\text{ess}} \sim (A_\eta)_{\text{ess}} (\textit{Mod rank one})$$

almost surely with respect to the product measure $\prod P \times \prod P$.

Anderson localization conjecture 1958

Discrete random Schrödinger operator on $l^2(\mathbb{Z}^d)$:

$$A_\omega = -\triangle + \sum_{m \in \mathbb{Z}^d} \omega_m(\,\cdot\,, \delta_m)\delta_m, \quad \text{where}$$

$$-\triangle f(x) = \sum_{m \in \mathbb{Z}^d, |m|=1} (f(x) - f(x+m)), \text{ and } \delta_m(x) = \left\{ \begin{array}{ll} 1 & x = m, \\ 0 & \text{else.} \end{array} \right\}$$

We let each random variable ω_i be i.i. distributed according to

$$dP = \chi_{[-c/2, c/2]}(x) dx / c.$$

Analytic results

- In 1958 P.W. Anderson suggested that sufficiently large impurities in a semi-conductor could lead to spatial localization of electrons.
- Here (Anderson localization = purely singular spectrum).
- Assume the scaling hypothesis.

Theorem (Fröhlich–Spencer CMP 1983)

In $d = 1$ localization occurs for all $c > 0$.

Theorem (Aizenmann–Molch. CMP '93, Schenker LMP '15)

In $d \geq 2$ localization occurs above dimension dependent threshold, e.g. $c_2 = 22.8$ and $c_3 = 100.6$.

The question of Anderson localization for small c in 2-D has been an open problem for over 50 years.

Finding cyclic vectors for random Hamiltonians

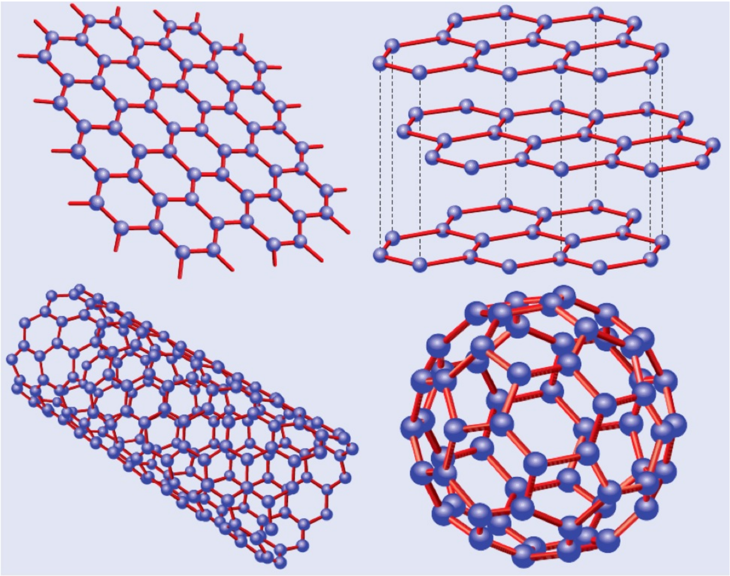
- Simon (Rev. MP '94): Assuming dense pure point spectrum on interval, all δ_m are cyclic for discr. rand. Schr. operator a.s.
- Jaksiç–Last (Duke '06): Class of cyclic evctors for $(A_\omega)_{\mathrm{s}}$ a.s.

Corollary (Abakumov–L–Poltoratski, JLMS 2013)

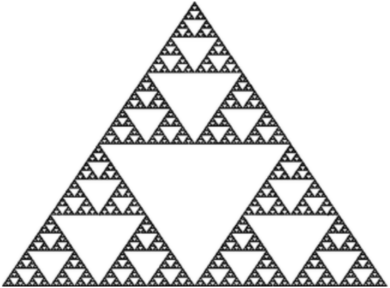
Consider $A_\omega = A + \sum_{m\in\mathbb{Z}^d} \omega_m(\,\cdot\,,\varphi_m)\varphi_m$ as above. If some φ_m is cyclic almost-surely, then any $\varphi \not\equiv 0$ is cyclic for A_ω almost surely.

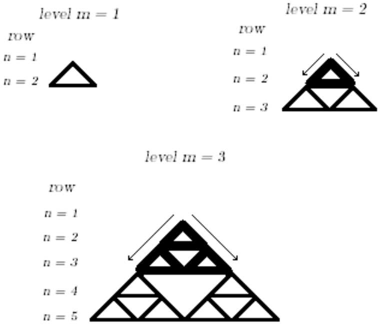
Method of cyclic vectors: Delocalization (i.e. $(A_\omega)_{\mathrm{ac}} \neq \mathbb{O}$) would be proven, if one can find some (non-zero) vector which is non-cyclic almost surely for the discr. rand. Schrödinger operator.
I have explored this approach to delocalization numerically.

Explore: Carbon allotropes



Explore: The Sierpinski gasket built from the top corner





Laplacian on fractals are the specialty of A. Teplyaev (U. Conn) and M. Stoiciu (Williams C.).

Rank One Perturbations in Analysis

- Nehari interpolation problem
- Holomorphic composition operators
- Rigid functions
- Poncelet's Theorem
- **Functional models (Nagy–Foiaș, deBr.–Rov., Nik.–Vasy.)**
- Carlesson embedding
- Two weight problem for Hilbert/Cauchy transform

Spectral representation & two-weight Hilbert transform

- Let μ be a positive finite Borel measure on $\mathbb{R}$.
- Consider family of self-adjoint rank 1 perturbations on $L^2(\mu)$:

$$A_\alpha = M_t + \alpha(\,\cdot\,,\mathbf{1})_{L^2(\mu)}\mathbf{1} \qquad \text{where } \alpha \in \mathbb{R}.$$

- By μ_α denote the spectral measure of A_α wrt function $\mathbf{1}$, i.e. for some unitary operator $U_\alpha : L^2(\mu) \to L^2(\mu_\alpha)$ we have

$$U_\alpha A_\alpha = M_s U_\alpha \quad \text{and} \quad U_\alpha \mathbf{1} = \mathbf{1}.$$

Theorem (L–Treil, JFA 2009)

Under the above assumptions, we have

$$U_\alpha f(s) = f(s) - \alpha \int_{\mathbb{R}} \frac{f(s) - f(t)}{s - t + i\varepsilon}\, d\mu(t) \qquad (1)$$

for all compactly supported C^1 functions f.

Rigidity: Vice versa, operators of the form (1) with trivial kernel are always spectral representations of rank one perturbations.

Explore: Boundedness of two-weight Hilbert transform

Let σ and ω be locally finite positive Borel measures on $\mathbb{R}$ without a common point mass. Consider the Hilbert transform $H_\sigma : L^2(\sigma) \to L^2(\omega)$ given by

$$H_\sigma f(x) = \text{p.v.} \int \frac{f(y) \mathrm{d}\sigma(y)}{y - x} .$$

Consider a Poisson extension of measure σ to upper half-plane

$$P(\sigma, I) = \int \frac{|I| \, \mathrm{d}\sigma(x)}{|I|^2 + \mathrm{dist}(x, I)^2}.$$

Theorem (Lacey, Duke 2014)

Hilbert transform $H_\sigma : L^2(\sigma) \to L^2(\omega)$ is bounded by M, iff uniformly over all intervals I we have $P(\sigma, I)P(\omega, I) \leq C$ and

$$\int_I |H_\sigma \chi_I|^2 d\omega \leq D\sigma(I), \qquad \int_I |H_\omega \chi_I|^2 d\sigma \leq D\omega(I).$$

For the best constants C and D above, we have $M \approx C^{1/2} + D$.

Explore: Matrix-valued Caldéron–Zygmund operators

- A (d-dim.) matrix weight W is a locally integrable function on $\mathbb{R}^N$ with values in positive-semidefinite $d \times d$ matrices.
- $L^2(W)$ is the space of all measurable $f : \mathbb{R}^N \to \mathbb{R}^d$ with

$$\|f\|^2_{L^2(W)} = \int (W(x)f(x), f(x))\mathrm{d}x < \infty.$$

- W satisfies matrix A_2 condition, if

$$[W]_{A_2} = \sup_Q \left| <W>_Q{}^{1/2} < W^{-1} >_Q^{1/2} \right|^2 < \infty.$$

- For $d = 1$, we have $\|H\|_{L^2(W)} \leq C[W]_{A_2}$.

Theorem (Nazarov–Petermichl–Treil–Volberg, Advances 2017)

For $d > 1$, we have $\|H\|_{L^2(W)} \leq C[W]^{3/2}_{A_2}$.

Proof uses that for every f there exists a uniformly sparse set of intervals $\mathcal{I}$ so that $|Tf| \leq C \sum_{I \in \mathcal{I}} < |f| >_I \chi_I$(pointwise).

Singular parts are mutually singular

- Consider a rank M perturbation $A_\alpha = A + \mathbf{B}\alpha\mathbf{B}^*$ of A, with $\mathbf{B} : \mathbb{C}^d \to \mathcal{H}$, cyclic $\operatorname{Ran}\mathbf{B}$ and symmetric $\alpha \in \mathbb{C}^{M\times M}$.
- Let μ_α be so that $\mathbf{B}^*(A_\alpha - \lambda\mathbf{I})^{-1}\mathbf{B} = \int_{\mathbb{R}} \frac{d\mu_\alpha(t)}{t-\lambda}$ $\forall\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Definition

Matrix-valued measures $\mu = W(\operatorname{tr}\mu)$ and $\nu = V(\operatorname{tr}\nu)$ are (vector) mutually singular ($\mu \perp \nu$) if one can extent W and V so that

$$\operatorname{Ran} W(t) \perp \operatorname{Ran} V(t) \qquad (\operatorname{tr}\mu)\text{-a.e. and } (\operatorname{tr}\nu)\text{-a.e.}$$

Theorem (L–Treil, JST 2021)

Singular parts of the matrix-valued measures μ and μ_α satisfy

$$\mu_{\mathrm{s}} \perp \alpha\mu_\alpha\alpha \qquad \textit{and} \qquad \mu_\alpha \perp \alpha\mu_{\mathrm{s}}\alpha.$$

The proof uses spectral representation and a matrix A_2 condition.

Summary: Perspectives on some problems in Analysis

Thank you!