Energy Estimates for a Multi-layer Shallow Water Wave Model with a Single Damped Layer

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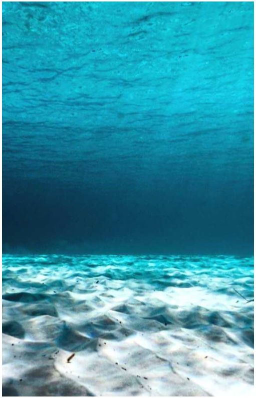
Motivational Speaking



Ocean tides, away from the coastlines, can be modeled with rotating shallow water wave equations (due to the length scales involved, the ocean is shallow!).

Tidal activity has strong impact on sediment transport and mixing of temperature and salinity. [1]

Calculating this solution is the goal of barotropic tide modelling, since the Earth's tides are assumed to have been occuring on a long enough time scale that memory of the initial conditions or past changes in topography are not relevant.

Current and Past Work - Layers

The work to be discussed in this presentation is part of a collaborative effort with Colin Cotter (Imperial College London), Jameson Graber, and Robert Kirby (Baylor University).

Prior work that this extends includes papers of Cotter and Kirby (2016) [2], and Cotter, Graber, Kirby, (2018) [1] which investigates modeling with a single equation with linear and non-linear drag, respectively. Here we investigate multiple layers with a single linearly damped layer, providing an energy estimate, and the key ideas of the proof along the way.

What is understood...

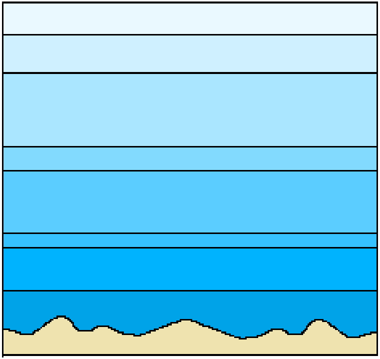
In the case of the single equation with linear damping, results of Cotter, Kirby show that in the absence of forcing, the damping drives the system energy exponentially to zero.

In the case of the single equation with *non-linear* damping, Cotter, Graber, Kirby show that

- initial energy must decay to zero
- rate depends on features of the nonlinearity (important: monotone, able to damp high velocity sufficiently)
- Error Estimate 1 optimal in mesh parameter, but exponential increase in time possible
- Error Estimate 2 uniform in time, but suboptimal in mesh parameter

Calculus + Sea

We consider N layers of fluid, the first layer with thickness D_1, the second D_2 down to D_N, the ith layer with density ρ_i.



The Model

Consider:

- N layers of fluid
- Thickness $D_i = D'_i + \bar{D}_i$ (Dynamic + At Rest)
- g gravity, f Coriolis
- ρ_i density in layer i

The model to be investigated is

$$\frac{\partial u_i}{\partial t} + f u_i^{\perp} + \underbrace{g\nabla\left(\sum_{i=1}^{N} D'_i + \sum_{j=1}^{i-1} \frac{\rho_j - \rho_i}{\rho_i} D'_j\right)}_{\text{layers on layers}} + d(|u_N|)\delta_{iN} u_N = 0$$

$$\frac{\partial D'_i}{\partial t} + \nabla \cdot (\bar{D}_i u_i) = 0,$$

with $u_i(x,t) : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$ the horizontal velocity.

Goal of the Talk

In what follows we will be setting up the framework and covering some of the ideas in proving the following result (here stated loosely):

Main Result

Under a suitable definition of an "energy", $E(t)$, we have that for all $T > 0$, there exists a constant C which depends only on the data, such that

$$\int_0^T E(t) \leq CE(0)$$

and further, $E(t)$ decays exponentially.

A Few Assumptions

$$\frac{\partial u_i}{\partial t} + f u_i^{\perp} + g\nabla \left(\sum_{j=1}^{N} D_j' + \sum_{j=1}^{i-1} \frac{\rho_j - \rho_i}{\rho_i} D_j' \right) + d(|u_N|)\delta_{iN} u_N = 0$$

$$\frac{\partial D_i'}{\partial t} + \nabla \cdot (\bar{D}_i u_i) = 0,$$

- Rigid Lid Assumption: $\sum_{j=1}^{N} D_j'$ is constant.
- Dampening Assumption: $d(|u_N|)$ constant with respect to $|u_N|$, nonnegative and bounded.

Simplifying the Task

$$\frac{\partial u_i}{\partial t} + fu_i^{\perp} + g\nabla\left(\sum_{j=1}^{N} D'_j + \sum_{j=1}^{i-1}\frac{\rho_j - \rho_i}{\rho_i}D'_j\right) + d(|u_N|)\delta_{iN}u_N = 0$$

$$\frac{\partial D'_i}{\partial t} + \nabla\cdot(\bar{D}_i u_i) = 0,$$

To recast the system as a matrix equation for ease of use, we make the following adjustments:

- Scale each equation for u_i by ρ_i, where $A_{ij} = \rho_{\min(i,j)}$

$$\rho_i\frac{\partial u_i}{\partial t} + \rho_i f u_i^{\perp} + g\nabla\left(\sum_{j=1}^{N} A_{ij}D'_j\right) + \rho_N d(|u_N|)\delta_{iN}u_N = 0$$

$$\frac{\partial D'_i}{\partial t} + \nabla\cdot(\bar{D}_i u_i) = 0,$$

Simplifying the Task

$$\rho_i \frac{\partial u_i}{\partial t} + \rho_i f u_i^{\perp} + g\nabla \left(\sum_{j=1}^{N} A_{ij} D_j' \right) + \rho_N d(|u_N|)\delta_{iN} u_N = 0$$

$$\frac{\partial D_i'}{\partial t} + \nabla \cdot (\bar{D}_i u_i) = 0,$$

- Introduce $\tilde{u}_i = \bar{D}_i u_i$ and $\mu_i = \bar{D}_i^{-1} \rho_i$

$$\mu_i \frac{\partial \tilde{u}_i}{\partial t} + \mu_i f \tilde{u}_i^{\perp} + g\nabla \left(\sum_{j=1}^{N} A_{ij} D_j' \right) + \rho_N d(\bar{D}_N^{-1} |\tilde{u}_N|)\delta_{iN} \tilde{u}_N = 0$$

$$\frac{\partial D_i'}{\partial t} + \nabla \cdot \tilde{u}_i = 0,$$

Simplifying the Task

$$\mu_i \frac{\partial \tilde{u}_i}{\partial t} + \mu_i f \tilde{u}_i^{\perp} + g\nabla \left(\sum_{j=1}^{N} A_{ij} D_j' \right) + \rho_N d(\bar{D}_N^{-1} |\tilde{u}_N|) \delta_{iN} \tilde{u}_N = 0$$

$$\frac{\partial D_i'}{\partial t} + \nabla \cdot \tilde{u}_i = 0,$$

- Let M be the diagonal matrix with $M_{ii} = \mu_i$ and B the matrix of all zeroes except for $B_{NN} = d$. Dropping the tildes and primes for simplicity, we have

$$M \frac{\partial u}{\partial t} + fMu^{\perp} + g\nabla(AD) + Bu = 0$$

$$\frac{\partial D}{\partial t} + \nabla \cdot u = 0.$$

Note: The gradients and divergences work componentwise.

Simplifying the Task

$$M\frac{\partial u}{\partial t} + fMu^{\perp} + g\nabla(AD) + Bu = 0$$
$$\frac{\partial D}{\partial t} + \nabla \cdot u = 0.$$

- Finally, the system admits a second-order primitive equation. Letting ϕ be such that $\frac{\partial \phi}{\partial t} = u$ and $\nabla \cdot \phi = D$, we have

$$M\partial_t^2 u + fM\partial_t \phi^{\perp} + B\partial_t \phi - gA\nabla(\nabla \cdot \phi) = 0.$$

The Coupling Matrix A and the Energy Functional E(t)

$$M\partial_t^2 u + fM\partial_t\phi^\perp + B\partial_t\phi - gA\nabla(\nabla\cdot\phi) = 0.$$

Recall that $A_{ij} = \rho_{\min(i,j)}$. This coupling matrix turns out (among other nice properties) to be symmetric positive definite, allowing for the following suitable energy definition:

$$E(t) = \frac{1}{2}\|\partial_t\phi\|_M^2 + \frac{1}{2}\|\nabla\cdot\phi\|_{gA}^2$$

Where $\|\psi\|_S^2 := \int_\Omega (S(x)\psi(x), \psi(x))\, \mathrm{d}x$.

Main Result Revisited

Theorem

With a suitable smallness assumptions on $\bar{D}_n$, for

$$E(t) = \frac{1}{2}\|\partial_t \phi\|_M^2 + \frac{1}{2}\|\nabla \cdot \phi\|_{gA}^2$$

there exists a constant C, depending only on initial data, such that for all $T > 0$,

$$\int_0^T E(t) \leq CE(0).$$

Further, for any fixed $T > C$, we have

$$E(t) \leq \frac{T}{C} e^{-\frac{\ln(T/C)}{T} t}$$

Proof Ideas: Coordinate Change via Lanczos

A has no zero entries, and so the coupling is intense! We will be making use of the Lanczos algorithm to affect a coordinate change. To justify this by assuring a basis is produced when seeded by the canonical vector e_n.

Proposition

For any $n \times n$ matrix B, if e_n is non orthogonal to all eigenvectors of B, then the eigenvalues are distinct.

Proof. Suppose v_1, v_2 are independent eigenvectors with common eigenvalue λ. Then $(v_2 \cdot e_n)v_1 - (v_1 \cdot e_n)v_2$ is an eigenvector of B that is orthogonal to e_n.

Proof Ideas: Coordinate Change via Lanczos

Proposition

Let B be any symmetric $n \times n$ matrix. If e_n is non orthogonal to all eigenvectors of B, then the set $\left\{e_n, Be_n, B^2 e_n, \ldots, B^{n-1} e_n\right\}$ forms a basis for $\mathbb{R}^n$.

Proof. Let v_i be an eigenbasis. Suppose $\sum_{i=1}^n c_i B^{i-1} e_n = 0$, then expanding around the eigenbasis and rearranging gives

$$0 = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} c_i \lambda_j^{i-1} \right) (e_n \cdot v_j) v_j.$$

Since $e_n \cdot v_j \neq 0$ for any j, it follows that $\sum_{i=1}^n c_i \lambda_j^{i-1} = 0$. Then the Vandermonde matrix V given by $V_{ij} = \lambda_i^{j-1}$ is singular. It is well known that this singularity implies repeated eigenvalues (a contradiction by the last proposition).

Proof Ideas: Coordinate Change via Lanczos

We now introduce the rescaling $\tilde{\psi} := M^{1/2}\phi$, followed by left multiplication of the model by $M^{-1/2}$,

$$\partial_t^2\tilde{\psi} + \partial_t\tilde{\psi}^\perp + \frac{d}{\mu_N}P_N\partial_t\tilde{\psi} - M^{-1/2}A\nabla(\nabla\cdot M^{-1/2}\tilde{\psi}) = 0$$

with weak form statement

$$(\partial_t^2\tilde{\psi} + \partial_t\tilde{\psi}^\perp + d\mu_N^{-1}P_N\partial_t\tilde{\psi}, \nu) + (\tilde{A}\nabla\cdot\psi, \nabla\cdot\nu) + (\tilde{A}\nabla\cdot\tilde{\psi}, P_N\tilde{\mu}\cdot\nu)$$
$$+ (\tilde{A}P_N\tilde{\mu}\cdot\tilde{\psi}, \nabla\cdot\nu) + (\tilde{A}P_N\tilde{\mu}\cdot\tilde{\psi}, P_N\tilde{\mu}\cdot\nu) = 0$$

Most importantly, $\tilde{A} := gM^{-1/2}AM^{-1/2}$ satisfies the previous two propositions, and thus Lanczos provides a matrix V such that $T = V^T\tilde{A}V$ is tridiagonal. Further, the rescaling keeps the first two terms simple.

Properties of tridiagonal T

The matrix T, which is symmetric tridiagonal by the Lanczos algorithm, has some additional surprising properties specific to the involvement of $\tilde{A}$ and our model, surprising especially due to the expectation of (x, y) dependence.

Lemma

Let $\alpha_i = T_{ii}$, $\beta_i = T_{i,i+1}$, Then for $2 \leq i \leq N$, α_i, β_i are positive constants. α_1, β_1 are positive, continuously differentiable functions satisfying the following. (The tilde eliminated now from $\tilde{A}$)

$$\frac{A_{NN}}{\mu_{N,1}} \leq \alpha_1 = \bar{D}_N \leq \frac{A_{NN}}{\mu_{N,0}}$$

$$\frac{b(M,A)}{\sqrt{\mu_{N,1}}} \leq \beta_1 = \frac{b(M,A)}{2\rho_N} \bar{D}_N^{-1/2} \nabla(\bar{D}_N) \leq \frac{b(M,A)}{\sqrt{\mu_{N,0}}}$$

$$\textit{where } b(M,A) = \sqrt{\mu_1^{-1} A_{N1}^2 + \cdots + \mu_{N-1}^{-1} A_{N(N-1)}^2}$$

Eliminating the Upper Diagonal

We alter the tridagonal T with elementary row operations by employing $\Gamma^T \nu$ as the weak form test function for a properly constructed matrix Γ. This yields two vital weak form statements. These are omitted from the slide for your viewing pleasure. The power of the statements lie in the recursive structure obtained from the altered T, allowing for estimates on ψ_i to be obtained inductively, depending only on $\psi_{i-1}, \psi_i, \psi_{i+1}$, and also allowing the base case for this induction.

This induction will be performed on an auxiliary energy.

An Auxiliary Energy

$$\tilde{E}(t) := \frac{1}{2}\|\partial_t \psi\|^2 + \frac{1}{2}\|\nabla \cdot \psi\|^2.$$

It can be shown that

$$\frac{1}{(1+2\eta)\max(\|T\|, 1)} E(t) \leq \tilde{E}(t) \leq \frac{\max(1, \|T^{-1}\|)}{1-\eta} E(t)$$

where η is a constant that depends only on $\bar{D}_n$, the domain (in the form of a Poincaré constant), and $\|T^{-1}\|$.

A Peek at the Induction

Using the weak form statement with test function ψ_1,

$$\sum_{j=1}^{n}(\gamma_{1,j}\partial_t^2\psi_j + \partial_t\psi_j^{\perp} + d\mu_n^{-1}\delta_{j1}\partial_t\psi_1, \psi_1) + (\tilde{\alpha}_1\nabla\cdot\psi_1, \nabla\cdot\psi_1 + \tilde{\mu}\cdot\psi_1)$$
$$+(\alpha_1\nabla\cdot\psi_1 + \beta_1\nabla\cdot\psi_2 + \alpha_1\tilde{\mu}\cdot\psi_1, \tilde{\mu}\cdot\psi_1) + (\tilde{\alpha}_1\tilde{\mu}\cdot\psi_1, \nabla\cdot\psi_1 + \tilde{\mu}\cdot\psi_1) = 0,$$

we obtain the base case estimate

$$\int_0^T \|\partial_t\psi_1\|^2 + \|\nabla\cdot\psi_1\|^2\,\mathrm{d}t \leq \frac{K_1}{\epsilon}E(0) + \epsilon\left(\sum_{j=2}^{N}\int_0^T \|\partial_t\psi_j\|^2 + \|\nabla\cdot\psi_j\|^2\mathrm{d}t\right)$$

A Peek at the Induction

The base case and $i = 2$ case require a bit of extra care, but once $i \geq 3$, we are safely into the three term recurrence relationship afforded by the coordinate change, and for $i \geq 3$,

$$\int_0^T \|\partial_t^2 \psi_i\|^2 + \|\nabla \cdot \psi_i\|^2 \mathrm{d}t \leq \frac{K_i}{\epsilon} E(0) + 2\epsilon \int_0^T \|\nabla \cdot \psi_{i+1}\|^2 \mathrm{d}t$$
$$+ \frac{L_i}{\epsilon^3} \int_0^T \|\partial_t \psi_{i-1}\|^2 + \|\nabla \cdot \psi_{i-1}\|^2 \mathrm{d}t + \frac{L_{i,*}}{\epsilon} \int_0^T \|\nabla \cdot \psi_{i-2}\|^2 \, \mathrm{d}t$$

Main Result Revisited 2: Again, The Sequel

Theorem

With a suitable smallness assumptions on $\bar{D}_n$ (to allow the induction), for

$$E(t) = \frac{1}{2}\|\partial_t \phi\|_M^2 + \frac{1}{2}\|\nabla \cdot \phi\|_{gA}^2$$

there exists a constant C, depending only on initial data, such that for all $T > 0$,

$$\int_0^T E(t) \leq CE(0).$$

Further, for any fixed $T > C$, we have exponential decay

$$E(t) \leq \frac{T}{C} e^{-\frac{\ln(T/C)}{T} t}$$

Linearity lending itself straight away for error estimates.

References

Thank you!

Colin J. Cotter, P. Jameson Graber, and Robert C. Kirby. "Mixed finite elements for global tide models with nonlinear damping". In: *Numerische Mathematik* 140.4 (2018), pp. 963–991. ISSN: 0945-3245. DOI: 10.1007/s00211-018-0980-4. URL: https://doi.org/10.1007/s00211-018-0980-4.

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