“Nodal Solutions for Neumann Problems with a Nonhomogeneous Differential Operator”

Michael E. Filippakis[1]

and

Nikolaos S. Papageorgiou[2]

[1]University of Piraeus, Department of Digital Systems, 80, Karaoli and Dimitriou Str, 18534, Piraeus,

[2]National Technical University, Department of Mathematics, Zografou Campus, Athens 157 80, Greece

1 Introduction

$$-\text{div}a(Du(z)) = f(z,u(z)) \text{ in } \Omega, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \quad (1)$$

- $\Omega \subseteq R^N$ bounded domain with smooth boundary,

- $a : R^N \to R^N$ is a continuous map which is C^1 on $R^N \setminus \{0\}$ and also it is strictly monotone. The precise hypotheses on $a(\cdot)$ are formulated in hypotheses $H(a)$.

- These hypotheses incorporate as special cases the p-Laplacian $(1 < p < \infty)$, the (p,q)-differential operator $(2 \leq q < p)$ and the generalized p-mean curvature operator $(2 \leq p < \infty)$.

- The reaction $f(z,x)$ is a Caratheodory function (i.e., for all $x \in R$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous) and exhibits p-linear growth near $\pm\infty$. In fact the precise hypotheses on the term $(f(z,x)$ make the energy functional of the problem coercive.

- Our aim is to prove a multiplicity theorem ("three solutions theorem") for problem (1), providing precise sign information for all the solutions.

- Multiplicity theorems for coercive Dirichlet problems driven by the p-Laplacian, were proved by Liu-Liu and Papageorgiou-Papageorgiou .

- For the Neumann problem, we have the works of Kyritsi-Papageorgiou and Motreanu-Papageorgiou. In Kyritsi-Papageorgiou, the left hand side of the equation has the form $-\Delta_p u+\beta\|u\|^{p-2}u$ with $\beta > 0$ (here $\Delta_p u = div(\|Du\|^{p-2}Du)$ for all $u \in W^{1,p}(\Omega)$ $1 < p < \infty$), while Motreanu-Papageorgiou has a nonhomogeneous differential operator that satisfies more restrictive condition, which exclude, for example the (p, q)-differential operator and the generalized p-mean curvature operator.

- None of the aforementioned works (Dirichlet and Neumann alike), provides sign information for all the solutions. To the best of our knowledge, our multiplicity theorem here, is the first result in the literature, establishing the existence of nodal solutions for Neumann problems driven by a nonhomogeneous differential operator.

- Note that the nonhomogeneity of the differential operator is the source of serious technical difficulties and the previous methods used to produce a nodal solution fail in this context (see, for example, the work of Aizicovici-Papageorgiou-Staicu). So, we need new conditions and techniques in order to guarantee the existence of a nodal solution.

- Our approach is variational based on the critical point theory, coupled with suitable truncation techniques and use of critical groups (Morse theory).

2 Mathematical Background

- Let X be a Banach space and X^* its dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X).
- Let $\varphi \in C^1(X)$. We say the φ satisfies the Cerami condition (the C-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded in X and $(1+\|x_n\|)\varphi'(x_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.
- This condition is more general than the usual in critical point theory "Palais Smale condition". It can be shown that the C-condition suffices to prove a deformation theorem and from it derive the minimax theory of the critical values of φ. In particular, we can state the following slightly more general version of the well-known "mountain pass theorem".

Theorem 2.1 *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$ and $r > 0$ satisfy*

$$\|x_1 - x_0\| > r \quad max\{\varphi(x_1), \varphi(x_0)\} < \inf[\varphi(x) : \|x - x_0\| = r] = \eta_r$$

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \geq 1} \varphi(\gamma(t))$$

$$\text{where } \Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0, \gamma(1) = x_1\},$$

then $c \geq \eta_r$ and c is a critical value of φ.

- In the analysis of problem (1) we will use the spaces $W^{1,p}(\Omega)$ and
$$C^1_n(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}.$$
- One can show that $W^{1,p}(\Omega) = \overline{C^1_n(\overline{\Omega})}^{\|\cdot\|}$ where $\|\cdot\|$ denotes the usual norm of the Sobolev space $W^{1,p}(\Omega)$
- We note that $C^1_n(\overline{\Omega})$ is an ordered Banach space with positive cone
$$C_+ = \{u \in C^1_n(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$
- This cone has a nonempty interior given by
$$intC_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

$\underline{H(a)}$: $a(y) = a_0(\|y\|)y$ for all $y \in R^N$, with $a_0(t) > 0$ for all $t > 0$ and

(i) $a \in C(R^N, R^N) \cap C^1(R^N\backslash\{0\}, R^N)$;

(ii) there exist $c_0 > 0$, $\eta \geq 0$ and $1 < p < \infty$ such that
$$c_0(\eta + \|y\|)^{p-2}\|\xi\|^2 \leq (\nabla a(y)\xi, \xi)_{R^N}$$
$$\text{for all } y \in R^N\backslash\{0\}, \text{ all } \xi \in R^N;$$

(iii) there exists $c_1 > 0$ such that
$$\|\nabla a(y)\| \leq c_1(\eta + \|y\|)^{p-2} \text{ for all } y \in R^N\backslash\{0\}$$
with $\eta \geq 0$ and $1 < p < \infty$ as in (ii);

Remark 2.1 *In what follows $G(\cdot)$ denotes the primitive of $a(\cdot)$, i.e.,*

$$G'(y) = a(y) \ \textit{for all} \ y \in R^N \ \textit{ and } \ G(0) = 0.$$

- The above hypotheses imply that $a(\cdot)$ is strongly monotone.
- Hence the primitive function $G(\cdot)$ is strictly convex.

Example 2.1 *The following maps satisfy hypotheses $H(a)$:*

$$a_1(y) = \|y\|^{p-2}y \ \textit{ with } \ 1 < p < \infty.$$

Map $a_1(\cdot)$ corresponds to the p-Laplace differential operator.

$$a_2(y) = \|y\|^{p-2}y + \mu\|y\|^{q-2}y \ \textit{ with } \ 2 \le q < p < \infty, \mu > 0.$$

Map $a_2(\cdot)$ corresponds to the (p, q)-differential operator.

$$a_3(y) = (1 + \|y\|^2)^{\frac{p-2}{2}}y \ \textit{ with } \ 2 \le p < \infty.$$

Map $a_3(\cdot)$ corresponds to the generalized p-mean curvature differential operator.

$$a_4(y) = \|y\|^{p-2}y(1 + \frac{1}{1 + \|y\|^p}) \ \textit{ with } \ 1 < p < \infty.$$

- Let $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle V(u), y\rangle = \int_\Omega (a(Du), Dy)_{R^N} dz \text{ for all } u, y \in W^{1,p}(\Omega). \tag{2}$$

Proposition 2.2 *If hypotheses $H(a)$ hold, then $V : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2) is continuous, strictly monotone (hence maximal monotone) and of type $(S)_+$ (i.e., if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$).*

- $f_0 : \Omega \times R \to R$ be a Caratheodory function such that

$$|f_0(z,x)| \leq a(z) + c|x|^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in R,$$

with $a \in L^\infty(\Omega)_+$, $c > 0$, $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N \end{cases}$.

- $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to R$ defined by

$$\varphi_0(u) = \int_\Omega G(Du(z))dz - \int_\Omega F_0(z,u(z))dz \text{ for all } u \in W^{1,p}(\Omega).$$

Proposition 2.3 *If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1_n(\overline{\Omega})$ minimizer of φ_0, i.e., there exists $r_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1_n(\overline{\Omega}),\ \|h\|_{C^1_n(\overline{\Omega})} \leq r_0,$$

then $u_0 \in C^{1,\beta}_n(\overline{\Omega})$ for some $\beta \in (0,1)$ and it is a local $W^{1,p}(\Omega)$-minimizer of φ_0, i.e., there exists $r_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega), \|h\| \leq r_1.$$

Remark 2.2 *In Motreanu-Papageorgiou $\eta = 0$ (see $H(a)(ii),(iii)$). However, a careful reading of their proof, reveals that it is not affected by assuming the more general hypotheses with $\eta \geq 0$. Moreover, in Motreanu-Papageorgiou a is z-dependent. We could have done the same thing here assuming that $a \in C(\overline{\Omega} \times R^N, R^N) \cap C^1(\overline{\Omega} \times (R^N \backslash \{0\}), R^N)$ and that the other hypotheses hold uniformly for all $z \in \overline{\Omega}$.*

For simplicity in the presentation, we have decided to work with an autonomous $a(\cdot)$.

Lemma 2.1 *If* $\eta \in L^{\infty}(\Omega)$, $\eta(z) \leq 0$ *a.e. on* Ω, $\eta \neq 0$, *then there exists* $\xi_0 > 0$ *such that*

$$\frac{c_0}{p(p-1)}\|Du\|_p^p - \int_{\Omega} \eta |u|^p dz \geq \xi_0 \|u\|^p \ \textit{for all} \ u \in W^{1,p}(\Omega).$$

<u>$H(f)$</u>: $f : \Omega \times R \to R$ is a Caratheodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leq a(z) + c|x|^{p-1}$ for a.a. $z \in \Omega$, all $x \in R$, with $a \in L^\infty(\Omega)_+, c > 0$;

(ii) there exists a function $\eta \in L^\infty(\Omega), \eta(z) \leq 0$ a.e. in Ω, $\eta \neq 0$ such that

$$\limsup_{x \to \pm\infty} \frac{pF(z,x)}{|x|^p} \leq \eta(z) \text{ uniformly for a.a. } z \in \Omega,$$

where $F(z,x) = \int_0^x f(z,s)ds$;

(iii) there exist $q \in (1,p)$ and $\delta_0 > 0$ such that

$$qF(z,x) \geq f(z,x)x > 0 \text{ for a.a. } z \in \Omega, \text{ all } 0 < |x| \leq \delta_0,$$

$$\underset{\Omega}{essinf}\, F(z, \pm\delta_0) > 0$$

and

$$f(z,x)x \geq c_4|x|^q - c_5|x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in R$$

with $c_4, c_5 > 0$;

(iv) for every $\rho > 0$ there exists $\xi_\rho > 0$ such that

$$f(z,x)x + \xi_\rho|x|^p \geq 0 \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho.$$

Example 2.2 *The following functions satisfy hypotheses $H(f)$:*

$$f_1(x) = |x|^{q-2}x - |x|^{p-2}x \quad \textit{with} \quad 1 < q < p < \infty$$

$$f_2(x) = \begin{cases} x - 2|x|^{r-2}x & \textit{if } |x| \leq 1 \\ -|x|^{p-2}x & \textit{if } |x| > 1 \end{cases} \quad \textit{with } 1 < r < \infty, \ 2 < p.$$

- X be a Banach space and $\varphi \in C^1(X)$, $C \in R$.
- $\varphi^c = \{x \in X : \varphi(x) \leq c\}$
- $K_\varphi = \{x \in X : \varphi'(x) = 0\}, K_\varphi^c = \{x \in K_\varphi : \varphi(x) = c\}$.
- Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homotopy group for the pair (Y_1, Y_2) with integer coefficients. The critical groups of φ at an isolated critical point $x_0 \in X$ of φ with $\varphi(x_0) = c$ (i.e., $x_0 \in K_\varphi^c$) are defined by

  $$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \backslash \{x_0\}) \text{ for all } k \geq 0,$$

  where U is a neighborhood of x_0 such that $K_\varphi \cap \varphi^c \cap U = \{x_0\}$. The excision property of singular homology implies that this definition is independent of the particular choice of the neighborhood U.
- Suppose that $\varphi \in C^1(X)$ satisfies the C-condition and that $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity are defined by

  $$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

- The second deformation theorem (see Gasinski-Papageorgiou (p.628)) implies that this definition is independent of the particular choice of the level $c < \inf \varphi(K_\varphi)$.

- Finally we mention that throughout this work, by $\|\cdot\|$ we will denote the norm of the Sobolev space. Already by $\|\cdot\|$ we have denoted also the norm of R^N. However, no confusion is possible, since it always be clear from the context which norm is used.

- For every $x \in R$, $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$.

- If $u \in W^{1,p}(\Omega)$, then $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that $u^{\pm} \in W^{1,p}(\Omega)$ and $u = u^+ - u^-$, $|u| = u^+ + u^-$.

- By $|\cdot|_N$ we denote the Lebesgue measure on R^N.

- Also, if $h : \Omega \times R \to R$ is measurable, then $N_h(u)(\cdot) = h(\cdot, u(\cdot))$ for all $u \in W^{1,p}(\Omega)$.

3 Constant Sign Solutions

In this section, first we establish the existence of at least two nontrivial constant sign smooth solutions (one positive and the other negative). Then we show that in fact we have "extremal" constant sign solutions, i.e., there exists a smallest nontrivial positive solution and a biggest nontrivial negative solution.

Proposition 3.1 *If hypotheses $H(a)$ and $H(f)$ hold, then problem (1) has at least two nontrivial constant sign smooth solutions*

$$u_0 \in intC_+ \ \ and \ \ v_0 \in -intC_+.$$

Proof:
We introduce the following truncations-perturbations of $f(z,\cdot)$:

$$\widehat{f}_+(z,x) = \begin{cases} 0 & \text{if } \ x \leq 0 \\ f(z,x) + x^{p-1} & \text{if } \ x > 0 \end{cases}$$

$$\widehat{f}_-(z,x) = \begin{cases} f(z,x) + |x|^{p-2}x & \text{if } \ x < 0 \\ 0 & \text{if } \ x \geq 0 \end{cases}.$$

- Both are Caratheodory functions.
- We set $\widehat{F}_\pm(z,x) = \int_0^x \widehat{f}_\pm(z,s)ds$ and introduce the C^1-functionals $\widehat{\varphi}_\pm : W^{1,p}(\Omega) \to R$ defined by

$$\widehat{\varphi}_\pm(u) = \int_\Omega G(Du(z))dz + \frac{1}{p}\|u\|_p^p - \int_\Omega \widehat{F}_\pm(z,u(z))dz \ \ \forall \ \ u \in W^{1,p}(\Omega).$$

- By virtue of hypotheses $H(f)(i),(ii)$, given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z,x) \leq \frac{1}{p}(\eta(z)+\varepsilon)x^p + c_\varepsilon \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{3}$$

- we infer that $\widehat{\varphi}_+$ is coercive.

- Also, exploiting the compact embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, we can easily check that $\widehat{\varphi}_+$ is sequentially weakly lower semicontinuous. Hence, by the Weierstrass theorem we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\widehat{\varphi}_+(u_0) = \inf[\widehat{\varphi}_+(u) : u \in W^{1,p}(\Omega)] = \widehat{m}_+. \tag{4}$$

- By virtue of hypotheses $H(f)(iii)$, $F(z,x) > 0$ for a.a. $z \in \Omega$, all $0 < |x| \leq \delta_0$. So, let $\xi \in (0, \delta_0]$. Then

$$\widehat{\varphi}_+(\xi) = -\int_\Omega F(z,\xi)dz < 0\ ,$$
$$\Rightarrow \widehat{\varphi}_+(u_0) = \widehat{m}_+ < 0 = \widehat{\varphi}_+(0), \text{ i.e., } u_0 \neq 0 \text{ (see (4))}.$$

¿From (4), we have

$$\widehat{\varphi}'_+(u_0) = 0$$
$$\Rightarrow V(u_0) + |u_0|^{p-2}u_0 = N_{\widehat{f}_+}(u_0). \tag{5}$$

- On (5) we act with $-u_0^- \in W^{1,p}(\Omega)$. Then

$$\frac{c_0}{p(p-1)}\|Du_0^-\|_p^p + \|u_0^-\|_p^p \leq 0$$
$$\Rightarrow u_0 \geq 0,\ u_0 \neq 0.$$

So, (5) becomes

$$V(u_0) = N_f(u_0),$$

$$\Rightarrow -diva(Du_0(z)) = f(z, u_0(z)) \text{ a.e. in } \Omega, \ \frac{\partial u_0}{\partial n} = 0 \text{ on } \partial\Omega \tag{6}$$

- From nonlinear regularity theory (see DiBenedetto (local regularity) and Lieberman (regularity up to the boundary)), we have $u_0 \in C_+ \backslash \{0\}$. Let $\rho = \|u_0\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $H(f)(iv)$. Then from (6) we have

$$-diva(Du_0(z)) + \xi_\rho (u_0(z)^{p-1} = f(z, u_0(z)) + \xi_\rho u_0(z)^{p-1} \geq 0 \text{ a.e. in } \Omega$$

$$\Rightarrow diva(Du_0(z)) \leq \xi_\rho u_0(z)^{p-1} \text{ a.e. in } \Omega,$$

$$\Rightarrow u_0 \in intC_+.$$

- Similarly, working with the C^1-functional $\widehat{\varphi}_-$, we produce another nontrivial constant sign smooth solution $v_0 \in -intC_+$.

- Next, we will produce extremal nontrivial constant sign smooth solutions (i.e., the smallest nontrivial positive solution and the biggest nontrivial negative solution).

- To this end, we consider the following nonlinear Neumann problem:

$$-diva(Du(z)) = f_0(z, u(z)) \text{ in } \Omega, \ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \quad (7)$$

$\underline{H(a)'}$: $a(y) = a_0(\|y\|)y$ $y \in R^N$ satisfies hypotheses $H(a)$ and

(iv) if $G_0(t) = \int_0^t a_0(s)sds,\ t > 0$, then $t \to G_0(t^{1/q})$ is convex with $q \in (1, p)$ as in $H(f)(iii)$

Example 3.1 *The maps* $a_1(y) = \|y\|^{p-2}y$ $,(1 < p < \infty)$, $a_2(y) = \|y\|^{p-2}y + \mu\|y\|^{\tau-2}y$ $,(2 \leq \tau < p < \infty)$ *and* $a_3(y) = (1 + \|y\|^2)^{\frac{p-2}{2}}y$ $,(2 \leq p < \infty)$ *all satisfy hypotheses* $H(a)'$.

$\underline{H_0}$: $f_0 : \Omega \times R \to R$ is a Caratheodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f_0(z, x)| \leq a(z) + c|x|^{r-1}$ for a.a.$z \in \Omega$, all $x \in R$, with $a \in L^\infty(\Omega)_+, c > 0$, $p \leq r < p^*$;

(ii) for a.a. $z \in \Omega$, $x \to \frac{f_0(z,x)}{x^{q-1}}$ is strictly decreasing on $(0, +\infty)$, $x \to \frac{f_0(z,x)}{|x|^{q-2}x}$ is strictly increasing on $(-\infty, 0)$ and $f_0(z, x)x \geq -\hat{c}|x|^p$ for all $x \in R$, with $\hat{c} > 0$.

Proposition 3.2 *If hypotheses* $H(a)'$ *and* H_0 *hold, then problem (7) has at most one nontrivial positive solution* $u_* \in intC_+$ *and at most one nontrivial negative solution* $v_* \in -intC_+$.

- Let $\sigma_+ : L^1(\Omega) \to \overline{R} = R \cup \{+\infty\}$ be the integral functional defined by

$$\sigma_+(u) = \begin{cases} \int_\Omega G(Du^{1/q})dz & \text{if } u \geq 0, u^{1/q} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases}.$$

- σ_+ is convex (see Diaz-Saa [**?**]), lower semicontinuous and it is not identically $+\infty$.

- Let u be a notrivial positive of (7).

- $$\sigma'_+(u^q)y = \frac{1}{q}\int_\Omega \frac{-diva(Du)}{u^{q-1}} ydz. \tag{8}$$

- Let v be another nontrivial positive solution of (7). Again we have $v \in intC_+$ and of course (8) holds with u replaced by v. Since $\sigma_+(\cdot)$ is convex, its Gateaux derivative is monotone and so we have

$$\int_\Omega (\frac{-diva(Du)}{u^{q-1}} + \frac{diva(Dv)}{v^{q-1}})(u^q - v^q)dz \geq 0,$$

$$\Rightarrow \int_\Omega (\frac{f_0(z,u)}{u^{q-1}} - \frac{f_0(z,v)}{v^{q-1}}(u^q - v^q)dz \geq 0. \tag{9}$$

- Hypothesis $H_0(ii)$ and (9) imply that $u = v$. This proves the uniqueness of the nontrivial positive solution (if it exists).

- Similarly, by considering

$$\sigma_-(u)_= \begin{cases} \int_\Omega G(D|u|^{1/q})dz & \text{if } u \leq 0, |u|^{1/q} \in W^{1,p}(\Omega) \\ +\infty & \text{otherwise} \end{cases}$$

We consider the following auxiliary Neumann problem:

$$\left\{ \begin{array}{l} -diva(Du(z)) = c_4|u(z)|^{q-2}u(z) - c_5|u(z)|^{p-2}u(z) \text{ in } \Omega \\ \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \end{array} \right\} \tag{10}$$

Proposition 3.3 *If hypotheses $H(a)$ hold, then problem (10) has a unique nontrivial positive solution $\overline{u} \in intC_+$ and by virtue of the oddness of (10) $-\overline{u} \in -intC_+$ is the unique nontrivial negative solution of (10).*

- We consider the C^1-functional $\psi_+ : W^{1,p}(\Omega) \to R$ defined by

  $$\psi_+(u) = \int_\Omega G(Du(z))dz + \frac{1}{p}\|u\|_p^p - \frac{c_4}{q}\|u^+\|_q^q + \frac{c_5}{p}\|u^+\|_p^p \;\forall\; u \in W^{1,p}(\Omega).$$

- We have

  $$\psi_+(u) \geq \frac{c_0}{p(p-1)}\|Du\|_p^p + \frac{1}{p}\|u^-\|_p^p + \frac{c_5}{p}\|u^+\|_p^p - \frac{c_4}{q}\|u^+\|_q^q$$
  $$\geq \frac{c_7}{p}\|u\|^p - \frac{c_4}{q}\|u\|^q \text{ for some } c_7 > 0.$$

- Since $q < p$, it follows that ψ_+ is coercive.
- Also ψ_+ is sequentially weakly lower semicontinuous.
- So, we can find $\overline{u} \in W^{1,p}(\Omega)$ such that

  $$\psi_+(\overline{u}) = \inf[\psi_+(u) : u \in W^{1,p}(\Omega)] = \overline{m}_+. \tag{11}$$

- Since $q < p$, for $\xi \in (0,1)$ small we have $\psi_+(\xi) < 0$ and so

  $$\psi_+(\overline{u}) = \overline{m}_+ < 0 = \psi_+(0), \text{ i.e. } \overline{u} \neq 0.$$

- From (11) we have

$$\psi_+'(\overline{u}) = 0,$$
$$\Rightarrow V(\overline{u}) = c_4(\overline{u}^+)^{q-1} - c_5(\overline{u}^+)^{p-1}. \tag{12}$$

- On (12) we act with $-\overline{u}^- \in W^{1,p}(\Omega)$ and obtain $\overline{u} \geq 0$, $\overline{u} \neq 0$. Hence (12) becomes

$$V(\overline{u}) = c_4\overline{u}^{q-1} - c_5\overline{u}^{p-1},$$

$$\Rightarrow -diva(D\overline{u}(z)) = c_4\overline{u}(z)^{q-1} - c_5\overline{u}(z)^{p-1} \text{ a.e. in } \Omega, \frac{\partial \overline{u}}{\partial n} = 0 \text{ on } \partial\Omega$$

(see Motreanu-Papageorgiou),

$$\Rightarrow diva(D\overline{u}(z)) \leq c_5\overline{u}(z)^{p-1} \text{ a.e. in } \Omega,$$

$\Rightarrow \overline{u} \in intC_+$ (see Montenegro (Theorem 6) and Pucci-Serrin).

- The uniqueness of the nontrivial positive solution $\overline{u} \in intC_+$ follows from Proposition 3.3.

- The oddness of (10) implies that $-\overline{u} \in intC_+$ is the unique nontrivial negative solution of (7).

- by $\mathcal{L}_+$ (resp. $\mathcal{L}_-$) we denote the set of nontrivial positive (resp. negative) solutions of (1). From Proposition 3.1 and its proof, we have

  $\mathcal{L}_+, \mathcal{L}_- \neq \emptyset$ and $\mathcal{L}_+ \subseteq intC_+, \mathcal{L}_- \subseteq -intC_+$.

Proposition 3.4 *If hypotheses $H(a)'$ and $H(f)$ hold and $\tilde{u} \in \mathcal{L}_+$ (resp. $\tilde{v} \in \mathcal{L}_-$), then $\overline{u} \leq \tilde{u}$ (resp. $\tilde{v} \leq -\overline{u}$.)*

- Let $\gamma_+ : \Omega \times R \to R$ be the Caratheodory function defined by

$$\gamma_+(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ c_4 x^{q-1} - (c_5 - 1)x^{p-1} & \text{if } 0 \leq x \leq \tilde{u}(z) \\ c_4 \tilde{u}(z)^{q-1} - (c_5 - 1)\tilde{u}(z)^{p-1} & \text{if } \tilde{u}(z) < x \end{cases} . \tag{13}$$

- Let $\Gamma_+(z,x) = \int_0^x \gamma_+(z,s)ds$ and consider the C^1-functional $\xi_+ : W^{1,p}(\Omega) \to R$ defined by

$$\xi_+(u) = \int_\Omega G(Du(z))dz + \frac{1}{p}\|u\|_p^p - \int_\Omega \Gamma_+(z,u(z))dz \ \forall \ u \in W^{1,p}(\Omega).$$

- From (13) it is clear that ξ is coercive. Also , it is sequentially weakly lower semicontinuous. Therefore by the Weierstrass theorem, we can find $\hat{y} \in W^{1,p}(\Omega)$ such that

$$\xi_+(\hat{y}) = \inf[\xi_+(u) : u \in W^{1,p}(\Omega)] = \widehat{m}_+. \tag{14}$$

- As before, exploiting the fact that $q < p$, we show that

$$\xi_+(\hat{y}) = \widehat{m}_+ < 0 = \xi_+(0), \quad \text{i.e. } \hat{y} \neq 0.$$

- From (14) we have

$$\xi_+'(\hat{y}) = 0,$$
$$\Rightarrow V(\hat{y}) + |\hat{y}|^{p-2}\hat{y} = N_{\gamma_+}(\hat{y}). \tag{15}$$

- On (15) we act with $-\hat{y}^- \in W^{1,p}(\Omega)$. We obtain $\hat{y} \geq 0$, $\hat{y} \neq 0$. Next, on (15) we act with $(\hat{y} - \tilde{u})^+ \in W^{1,p}(\Omega)$.

- So, we have proved that

$$\hat{y} \in [0, \tilde{u}] = \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \tilde{u}(z) \ \text{ a.e. in } \ \Omega\}.$$

- Hence, (15) becomes

$$V(\hat{y}) = c_4 \hat{y}^{q-1} - c_5 \hat{y}^{p-1} \ \ \text{(see (13))},$$
$$\Rightarrow \hat{y} = \overline{u} \in intC_+ \ \ \text{(see Proposition 3.3)},$$
$$\Rightarrow \overline{u} \leq \tilde{u}.$$

  Similarly, if $\tilde{v} \in \mathcal{L}_- \subseteq -intC_+$, then as above we show that $\tilde{v} \leq -\overline{u}$.

- Using Proposition 3.4, we show the existence of extremal nontrivial constant sign solutions of (1).

Proposition 3.5 *If hypotheses $H(a)'$ and $H(f)$ hold, then problem (1) has a smallest nontrivial positive solution $u_+ \in intC_+$ and a biggest nontrivial negative solution $v_- \in -intC_+$.*

- Let $C \subseteq \mathcal{L}_+$ be a chain (i.e., a totally ordered subset of $\mathcal{L}_+$). From Dunford-Schwartz (p.336), we know that we can find $\{u_n\}_{n\geq 1} \subseteq C$ such that

$$infC = \inf_{n\geq 1} u_n.$$

- Since C is totally ordered, by virtue of Lemma 1.1.5 (p.15) of Heikkila-Lakshmikantham, we may assume that $\{u_n\}_{n\geq 1}$ is decreasing. We have

$$V(u_n) = N_f(u_n),\ \overline{u} \leq u_n \leq u_1, \text{ for all } n \geq 1 \text{ (see Proposition 3.4)},$$

$$\Rightarrow \{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{16}$$

- We may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^p(\Omega) \text{ as } n \to \infty. \tag{17}$$

- On (16) we act with $u_n - u$, pass to the limit as $n \to \infty$ and use (17). Then

$$\lim_{n\to\infty} \langle V(u_n), u_n - u\rangle = 0,$$

$$\Rightarrow u_n \to u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 2.2).} \tag{18}$$

- So, if in (16) we pass to the limit as $n \to \infty$, then

$$V(u) = N_f(u),\ \ \overline{u} \leq u,$$

$$\Rightarrow u \in intC_+ \text{ solves (1) (nonlinear regularity), i.e., } u \in \mathcal{L}_+.$$

- Since $u = infC$ and C was an arbitrary chain in $\mathcal{L}_+$, from the Kuratowski-Zorn lemma, we infer that $\mathcal{L}_+$ has a minimal element. Since $a(\cdot)$ is monotone, as in Aizicovici-Papageorgiou-Staicu (Lemma 1), we show that $\mathcal{L}_+$ is downward directed, i.e., if $u, u' \in \mathcal{L}_+$, then we can find $y \in \mathcal{L}_+$ such that $y \leq min\{u, u'\}$. So, if $u \in \mathcal{L}_+$, then we can find $y \in \mathcal{L}_+$ such that $y \leq min\{u_+, u\}$ and so by the minimality of u_+, we infer that $y = u_+ \leq u$, hence u_+ is the smallest nontrivial positive solution of (1).
- Similarly, since $\mathcal{L}_-$ is upward directed, i.e., if $v, v' \in \mathcal{L}_-$, then we can find $y \in \mathcal{L}_-$ such that $max\{v, v'\} \leq y$ (see Aizicovici-Papageorgiou-Staicu, Lemma 2), as above using the Kuratowski-Zorn lemma, we can find $v_- \in -intC_+$ the biggest negative solution of (1).

4 Nodal solutions

In this section, using the extremal nontrivial constant sign solutions of (1) produced in Proposition 3.5, we will generate a nodal (sign changing) solution.

Proposition 4.1 *If hypotheses $H(a)'$ and $H(f)$ hold, then problem (1) has a nodal solution $y_0 \in C^1(\overline{\Omega})$.*

- Let $u_+ \in intC_+$ and $v_- \in -intC_+$ be the two extremal nontrivial constant sign solutions of (1) produced in Proposition 3.5. We introduce the following truncation-perturbation of the reaction term

$$h(z,x) = \begin{cases} f(z, v_-(z)) + |v_-(z)|^{p-2}v_-(z) & \text{if } \ x < v_-(z) \\ f(z,x) + x^{p-1} & \text{if } \ v_-(z) \le x \le u_+(z) \\ f(z, u_+(z)) + u_+(z)^{p-1} & \text{if } \ u_+(z) < x. \end{cases} \quad (19)$$

- This is a Caratheodory function.
- We set $H(z,x) = \int_0^x h(z,s)ds$ and consider the C^1-functional $\psi : W^{1,p}(\Omega) \to R$ defined by

$$\psi(u) = \int_\Omega G(Du(z))dz + \frac{1}{p}\|u\|_p^p - \int_\Omega H(z,u(z))dz \ \ \forall \ \ u \in W^{1,p}(\Omega).$$

- Also, let $h_\pm(z,x) = h(z, \pm x^\pm)$, $H_\pm(z,x) = \int_0^x h_\pm(z,s)ds$ and consider the C^1-functionals $\psi_\pm : W^{1,p}(\Omega) \to R$ defined by

$$\psi_\pm(u) = \int_\Omega G(Du(z))dz + \frac{1}{p}\|u\|_p^p - \int_\Omega H_\pm(z,u(z))dz \ \text{ for all } \ u \in W^{1,p}$$

- As before (see the proof of Proposition 3.4), we can show that

$$K_\psi \subseteq [v_-, u_+],\ K_{\psi_+} \subseteq [0, u_+],\ K_{\psi_-} \subseteq [v_-, 0],\ \text{(see (19))}.$$

- by virtue of the extremality of u_+ and v_- (see Proposition 3.5), we have

$$K_\psi \subseteq [v_-, u_+],\ K_{\psi_+} = \{0, u_+\},\ K_{\psi_-} = \{v_-, 0\}. \quad (20)$$

- Claim: u_+ and v_- are local minimizers of ψ.

- Clearly ψ_+ is coercive

- Also, it is sequentially weakly lower semicontinuous.

- So, we can find $\hat{u} \in W^{1,p}(\Omega)$ such that

$$\psi_+(\hat{u}) = inf[\psi_+(u) : u \in W^{1,p}(\Omega)] < 0 = \psi_+(0)\ \ \text{(see } H(f)(iii)\text{)},$$

$$\Rightarrow \hat{u} = u_+\ \ \text{(see (20))}.$$

- Note that $\psi|_{C_+} = \psi_+|_{C_+}$ and recall that $u_+ \in intC_+$. So, we infer that u_+ is a local $C^1_n(\overline{\Omega})$-minimizer of ψ, hence by Proposition 2.3 u_+ is also a local $W^{1,p}(\Omega)$-minimizer of ψ. Similarly for $v_- \in -intC_+$. This proves the Claim.

- We may assume that $\psi(v_-) \leq \psi(u_+)$ and by virtue of the Claim, as in Aizicovici-Papageorgiou-Staicu , we can find $\rho \in (0, 1)$ small such that $\|u_+ - v_-\| > p$ and

$$\psi(v_-) \leq \psi(u_+) < inf[\psi(u) : \|u - u_+\| = \rho] = \eta_\rho. \quad (21)$$

- Evidently ψ is coercive (see (19)). So, it satisfies the C-condition. This fact and (21) permit the use of Theorem 2.1 (the mountain pass theorem). Hence we can find $y_0 \in W^{1,p}(\Omega)$ such that

$$y_0 \in K_\psi \subseteq [v_-, u_+] \text{ (see (20)) and } \eta_\rho \leq \psi(y_0) \text{ (see (21)).} \tag{22}$$

- From (19), (21) and (22) we infer that y_0 solves (1), $y_0 \notin \{u_+, v_-\}$ and $y_0 \in C^1_n(\overline{\Omega})$ (nonlinear regularity). Since y_0 is a critical point of ψ of mountain pass type, we have

$$C_1(\psi, y_0) \neq 0 \text{ (see Chang (p.89)).} \tag{23}$$

- On the other hand, hypothesis $H(f)(iii)$ implies that

$$C_k(\psi, 0) = 0 \text{ for all } k \geq 0 \tag{24}$$

- (this is proved in Jiu-Su (Proposition 2.1) and Mugnai and it is based on the fact that hypothesis $H(f)(iii)$ implies that $F(z,x) \geq \tilde{c}|x|^q$ for a.a. $z \in \Omega$ all $|x| \leq \delta_0$ and for some $\tilde{c} > 0$). Comparing (23) and (24), we conclude that $y_0 \neq 0$. Since $y_0 \in K_\psi \backslash \{0, u_+, v_-\} \subseteq [v_-, u_+] \backslash \{0, u_+, v_-\}$ (see (20)) and by virtue of the extremality of u_+, v_- (see Proposition 3.5), we conclude that $y_0 \in C^1_n(\overline{\Omega})$ (nonlinear regularity) is nodal.

Therefore, we can state the following multiplicity theorem for problem (1).

Theorem 4.2 *If hypotheses $H(a)$ and $H(f)$ hold, then problem (1) has at least three nontrivial smooth solutions*

$$u_0 \in intc_+, v_0 \in -intC_+ \ \ and \ \ y_0 \in C_n^1(\overline{\Omega}) \ \ nodal;$$

moreover, problem (1) has extremal nontrivial constant sign solutions.