Flux based finite element methods

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Model problem

$$\begin{aligned} Lu = -\nabla \cdot \mathcal{A}\nabla u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Goal: approximation of the flux $\boldsymbol{\sigma} = -\mathcal{A}\nabla u$ (and more)

Finite Element methods

1. Standard Galerkin method.(post-processing for the flux variable)

$$(\mathcal{A}\nabla u\,,\,\nabla v) = (f, v)$$

2. Mixed Galerkin method.(more degrees of freedom) (A. Alonso, C. Carstenson, A. Demlow, M. Fortin, R. Hoppe, R. Raviart, J. Thomas)

$$\begin{aligned}(\mathcal{A}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) - (\nabla\cdot\boldsymbol{\tau}, u) &= 0,\\ (\nabla\cdot\boldsymbol{\sigma}, v) &= (f, v)\end{aligned}$$

3. Least-squares Finite Element method.(more degree of freedom) (Z. Cai, G, Carey, R. Lazarov, T. Manteuffel, S. McCormick, A. Pehlivanov, P. Vassilevski)

Model problem

$$\begin{aligned} Lu = -\nabla \cdot \mathcal{A}\nabla u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

First-order system

$$\begin{aligned} \boldsymbol{\sigma} + \mathcal{A}\nabla u &= 0 \text{ on } \Omega, \\ \nabla \cdot \boldsymbol{\sigma} &= f \text{ on } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Solution spaces

$$\begin{aligned} u &\in V := H_0^1(\Omega), \\ \boldsymbol{\sigma} &\in W := H(\text{div}) = \{\boldsymbol{\sigma} \in L^2 ; \nabla \cdot \boldsymbol{\sigma} \in L^2\}. \end{aligned}$$

Let $\mathbf{X} = V \times W$

Least-squares finite element method

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + (\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau} + \mathcal{A}\nabla v) = (f, \nabla \cdot \boldsymbol{\tau})$$

Least-squares finite element method

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + (\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau} + \mathcal{A}\nabla v) = (f, \nabla \cdot \boldsymbol{\tau})$$

Introduce a weight δ, (small such as h^2)

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + \delta(\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau} + \mathcal{A}\nabla v) = (f, \nabla \cdot \boldsymbol{\tau})$$

Least-squares finite element method

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + (\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau} + \mathcal{A}\nabla v) = (f, \nabla \cdot \boldsymbol{\tau})$$

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Take v=0, then

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + \delta(\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau}) = (f, \nabla \cdot \boldsymbol{\tau})$$

Least-squares finite element method

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Introduce a weight δ, (small such as h^2)

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + \delta(\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau} + \mathcal{A}\nabla v) = (f, \nabla \cdot \boldsymbol{\tau})$$

Take v=0, then

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + \delta(\mathcal{A}^{-1}(\boldsymbol{\sigma} + \mathcal{A}\nabla u), \boldsymbol{\tau}) = (f, \nabla \cdot \boldsymbol{\tau})$$

Move $\delta(\nabla u, \boldsymbol{\tau})$ to the right and use integration by parts,

$$(\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + \delta(\mathcal{A}^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau}) = (f, \nabla \cdot \boldsymbol{\tau}) + \delta(u, \nabla \cdot \boldsymbol{\tau}).$$

Reduced (Hybrid) Finite Element Method

Step 1 (Coarse-grid solution) On a coarse mesh $\mathcal{T}_H$, obtain the standard Galerkin solution u_H^G satisfying

$$(\mathcal{A}\nabla u_H^G, \nabla v_H) = (f, v_H) \qquad \text{for all } v_H \in V_H^r. \tag{1}$$

Step 2 (Fine-grid solution) On a finer mesh $\mathcal{T}_h$, find the $H(\text{div})$ projection $\boldsymbol{\sigma}_h \in W_h$ for the given data $f + u_H^G$, i.e.

$$(\nabla\cdot\boldsymbol{\sigma}_h, \nabla\cdot\boldsymbol{\tau}_h) + \delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = (f + \delta u_H^G, \nabla\cdot\boldsymbol{\tau}_h) \qquad \text{for all } \boldsymbol{\tau}_h \in W_h^k. \tag{2}$$

Quasi-Orthogonality

$$(\nabla\cdot(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), \nabla\cdot\boldsymbol{\tau}_h) + \delta(\mathcal{A}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h) = (\delta(u - u_H^G), \nabla\cdot\boldsymbol{\tau}_h), \tag{3}$$

for all $\boldsymbol{\tau}_h \in W_h^k$.

Some of the advantages of New method

- Well-developed fast solver on fine grid (D. Arnold, R. Ralk, R. Winther and R. Hiptmair, J. Xu)

- Smaller problem size.

- elimination of the need for artificial stabilization techniques (no *inf-suf* condition.)

- a practical and sharp *a posteriori* error estimator

Mixed methods

$$
\begin{aligned}
(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) - (\nabla \cdot \boldsymbol{\tau}_h, u_h^m) &= 0, \\
(\nabla \cdot \boldsymbol{\sigma}_h^m, v_h) &= (f, v_h)
\end{aligned}
$$

$$
\begin{aligned}
\delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) - \delta(\nabla \cdot \boldsymbol{\tau}_h, u_h^m) &= 0, \\
(\nabla \cdot \boldsymbol{\sigma}_h^m, v_h) &= (f, v_h)
\end{aligned}
$$

Taking $v_h = \nabla \cdot \boldsymbol{\tau}_h$,

$$
(\nabla \cdot \boldsymbol{\sigma}_h^m, \nabla \cdot \boldsymbol{\tau}_h) + \delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m, \boldsymbol{\tau}_h) = (f + \delta u_h^m . \nabla \cdot \boldsymbol{\tau}_h).
$$

Mixed method

$$(\nabla\cdot\boldsymbol{\sigma}_h^m,\nabla\cdot\boldsymbol{\tau}_h)+\delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^m,\boldsymbol{\tau}_h)=(f+\delta u_h^m.\nabla\cdot\boldsymbol{\tau}_h).$$

Reduce method (Hybrid method)

$$(\nabla\cdot\boldsymbol{\sigma}_h,\nabla\cdot\boldsymbol{\tau}_h)+\delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h)=(f+\delta u_H^G,\nabla\cdot\boldsymbol{\tau}_h).$$

$$(\nabla\cdot(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h),\nabla\cdot\boldsymbol{\tau}_h)+\delta(\mathcal{A}^{-1}(\boldsymbol{\sigma}_h^m-\boldsymbol{\sigma}_h),\boldsymbol{\tau}_h)=\delta(u_h^m-u_H^G,\nabla\cdot\boldsymbol{\tau}_h).$$

Supercloseness property 1.

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^m\|_{H(\mathsf{div})} \le C\sqrt{\delta}\Big(H\|u - u_H^G\|_{W_2^1(\Omega)} + Ch\|\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}\|_{H(\mathsf{div})}\Big).$$

Basic Error Estimate (2017, Ku, Lee, Sheen)

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le C\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_0+\sqrt{\delta}(h\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{H(\mathrm{div})}+CH\|u-u_H^G\|_{W_2^1(\Omega)}).$$

Basic Error Estimate (RT space of order 0)

$$\begin{aligned}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{L_2(\Omega)} &\le C\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{L_2(\Omega)} \\ &\quad +\sqrt{\delta}h\|\boldsymbol{\sigma}-\Pi_h\boldsymbol{\sigma}\|_{H(\mathrm{div})}+C\sqrt{\delta}H\|u-u_H^G\|_{W_2^1(\Omega)} \\ &\le Ch\|\boldsymbol{\sigma}\|_1+Ch^2\|\nabla\cdot\boldsymbol{\sigma}\|_1+C\sqrt{\delta}H^2\|u\|_2.\end{aligned}$$

Relation between H and h

$$h=\sqrt{\delta}H^2.$$

Numerical example

$$
\begin{aligned}
-\triangle u &= f \qquad \text{in } \Omega, \\
u &= 0 \qquad \text{on } \partial\Omega,
\end{aligned}
$$

where $\Omega = [0, 1] \times [0, 1]$ and $u = (x^2 - x)(y^2 - y)$.

Results with fixed coarse mesh with $h = H^2$ and $\delta = 1$

$1/H$	$\lvert u - u_H^G \rvert_1$	$1/h$	$\lvert u - u_h \rvert_1$	$\lVert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rVert_0$	Rate	$\lVert \boldsymbol{\sigma}_h + \nabla u_H^G \rVert_0$	$\frac{\lVert \boldsymbol{\sigma}_h + \nabla u_H^G \rVert_0}{\lvert u - u_H^G \rvert_1}$
4	0.549D-01	16	0.143D-01	0.931D-02	x	0.551D-01	1.00
8	0.285D-01	64	0.358D-02	0.234D-02	1.00	0.285D-01	1.00
16	0.143D-01	256	0.895D-03	0.585D-03	1.00	0.143D-01	1.00
32	0.716D-02	1024	0.224D-03	0.146D-03	1.00	0.716D-02	1.00

Results with fixed coarse mesh with $H = 1/4$ and $\delta = h^2$

$1/H$	$\lvert u - u_H^G \rvert_1$	$1/h$	$\lvert u - u_h \rvert_1$	$\lVert \boldsymbol{\sigma} - \boldsymbol{\sigma}_h \rVert_0$	Rate	$\lVert \boldsymbol{\sigma}_h + \nabla u_H^G \rVert_0$	$\frac{\lVert \boldsymbol{\sigma}_h + \nabla u_H^G \rVert_0}{\lvert u - u_H^G \rvert_1}$
4	0.549D-01	16	0.143D-01	0.932D-02	x	0.551D-01	1.00
4	0.549D-01	64	0.358D-02	0.233D-02	1.00	0.549D-01	1.00
4	0.549D-01	256	0.895D-03	0.582D-03	1.00	0.549D-01	1.00
4	0.549D-01	1024	0.224D-03	0.146D-03	1.00	0.549D-01	1.00

Comparison with mixed method

$1/H$	$\|u - u_H^G\|_0$	$1/h$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0$	DOF	$\|u - u_h^M\|_0$	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^M\|_0$	DOF
4	0.501D-02	16	0.931D-02	841	0.219D-02	0.928D-02	1312
8	0.132D-02	64	0.234D-02	12705	0.549D-03	0.233D-02	20608
16	0.332D-03	256	0.585D-03	198577	0.137E-03	0.582D-03	328192

From two–grids to one–grid

Basic Error Estimate (RT space of order 0)

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le Ch\|\boldsymbol{\sigma}\|_1 + Ch^2\|\nabla \cdot \boldsymbol{\sigma}\|_1 + C\sqrt{\delta}H\|u - u_H^G\|_1.$$

Relation between H and h

$$h = \sqrt{\delta}H^2.$$

take $u_H^G = 0$ and $\delta = h^2$

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \le Ch\|\boldsymbol{\sigma}\|_1 + Ch^2\|\nabla \cdot \boldsymbol{\sigma}\|_1 + C\sqrt{\delta}\|u\|_1.$$

Find $\boldsymbol{\sigma}^0$ defied by

$$(\nabla \cdot \boldsymbol{\sigma}_h^0, \nabla \cdot \boldsymbol{\tau}_h) + \delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^0, \boldsymbol{\tau}_h) = (f, \nabla \cdot \boldsymbol{\tau}_h), \text{ (no } \delta(u_G^H, \nabla \cdot \boldsymbol{\tau}_h))$$

then, for $n = 1, 2, ...$ find $\boldsymbol{\sigma}_h^n$ defined by

$$(\nabla \cdot \boldsymbol{\sigma}_h^n, \nabla \cdot \boldsymbol{\tau}_h) + \delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h) = (f, \nabla \cdot \boldsymbol{\tau}_h) + \delta(A^{-1}\boldsymbol{\sigma}_h^{n-1}, \boldsymbol{\tau}_h).$$

$$\boldsymbol{\sigma}_h^n \to \boldsymbol{\sigma}_h^m, \text{ as } n \to \infty.$$

u_h can be recovered by

$$(u_h, \nabla \cdot \boldsymbol{\tau}) = (\mathcal{A}^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)$$

with error estimate

$$\|u - u_h\|_0 \le Ch\|u\|_2 + \sqrt{\delta}\|u_h^m\|_0.$$

with $u_H^G = 0$ **and** $\delta = h^2$

$DOFs$	h	$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$	Rate	$\|u - u_h\|$	Rate
56	2.50e-01	6.37378e-01	x	1.50786e-01	x
208	1.25e-01	3.24488e-01	0.9740	7.82612e-02	0.9461
800	6.25e-02	1.63279e-01	0.9908	3.94786e-02	0.9872
3136	3.12e-02	8.18073e-02	0.9970	1.97822e-02	0.9969
12416	1.56e-02	4.09294e-02	0.9991	9.89644e-03	0.9992
49408	7.81e-03	2.04685e-02	0.9997	4.94889e-03	0.9998

A posteriori error estimators

Take $\mathcal{A} = Identity$ for simplicity.

Assume

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L_2(\Omega)} \leq m\|\nabla(u - u_H^G)\|_{L_2(\Omega)}, \tag{4}$$

where $0 \leq m < 1$.

$\|\boldsymbol{\sigma}_h + \nabla u_H^G\|$ as an estimator for $\|\nabla(u - u_H^G)\|_0$

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h + \nabla u_H^G\|_0 &= \|(\boldsymbol{\sigma}_h - \boldsymbol{\sigma} - \nabla u + \nabla u_H^G)\|_0 \\
&\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\nabla u - \nabla u_H^G\|_0 \\
&\leq (1+m)\|\nabla u - \nabla u_H^G\|_0
\end{aligned}
$$

Thus,

$$
\frac{1}{1+m}\|\boldsymbol{\sigma}_h + \nabla u_H^G\|_0 \leq \|\nabla u - \nabla u_H^G\|_0.
$$

$$
\begin{aligned}
\|\nabla u - \nabla u_H^G\|_0 &\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h + \nabla u - \nabla u_H^G\|_0 \\
&\qquad + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \\
&\leq \|\boldsymbol{\sigma}_h + \nabla u_H^G\|_0 + m\|\nabla u - \nabla u_H^G\|_0.
\end{aligned}
$$

Thus,

$$
\|\nabla u - \nabla u_H^G\|_0 \leq \frac{1}{1-m}\|\boldsymbol{\sigma}_h + \nabla u_H^G\|_0.
$$

Equivalent a posteriori error estimator

$$\frac{1}{1+m}\|\boldsymbol{\sigma}_h + \nabla u_H^G\|_0 \le \|\nabla u - \nabla u_h\|_0 \le \frac{1}{1-m}\|\boldsymbol{\sigma}_h + \nabla u_H^G\|_0.$$

A posteriori error estimates for the flux

$$J_l = \begin{cases} [[\boldsymbol{\sigma}_h \cdot t]] & \text{if } l \not\subset \partial\Omega, \\ 2(\boldsymbol{\sigma}_h \cdot t) & \text{if } l \subset \partial\Omega. \end{cases}$$

Then, for any $T \in \mathcal{T}_H$, we define

$$\eta^2(T) = |T| \|\text{rot}\boldsymbol{\sigma}_h\|_{0,T}^2 + \frac{1}{2} \sum_{l \subset \partial T} |l| \|J_l\|_{0,l}^2,$$

where $|T|$ and $|l|$ are the area of T and the length of l, respectively, and let

$$\eta = (\sum_{T \in \mathcal{T}_h} \eta_T^2)^{1/2}.$$

$$
\begin{aligned}
c\,\eta(T) &\leq \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,N(T)}, \\
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 &\leq C\Big(\sum_{T\in\mathcal{T}_h} \eta_T^2 + h^2\|f - P_h f\|_0^2 + \delta^2\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_H\|_0^2\Big)^{1/2}.
\end{aligned}
$$

Adaptive procedure

$$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}.$$

$$(\nabla \cdot \boldsymbol{\sigma}_h, \nabla \cdot \boldsymbol{\tau}_h) + \delta(\mathcal{A}^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = (f, \nabla \cdot \boldsymbol{\tau}_h) + \delta(A^{-1}\boldsymbol{\sigma}_H, \boldsymbol{\tau}_h).$$

With mild conditions (small δ and initial mesh size), we have $\rho + \gamma < 1$ such that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2 \leq \rho\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_H\|^2 + \gamma\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{HH}\|^2 + \text{osc}(H),$$

where

$$\text{osc}(H) = \|H(f - P_H f)\|^2.$$

Efficient solver

The Raviart-Thomas space of index r is given by

$$\mathbf{V}_h = \{\mathbf{v} \in H(\mathrm{div}) : \mathbf{v}|_T \in P_r(T) + (x,y)P_r(T) \text{ for all } T \in \mathcal{T}_h\}.$$

Here $P_r(T)$ denotes the set of polynomial functions of degree at most r on T.

Decomposition of $\mathbf{V}_h$.

$$W_h = \{s \in H^1 : s|_T \in P_{r+1}(T)\}, \quad S_h = \{q \in L_2 : q|_T \in P_r(T)\}$$

Discrete gradient operator $\mathbf{grad}_h : S_h \to \mathbf{V}_h$ defined by

$$(\mathbf{grad}_h q, \mathbf{v}) = -(q, \nabla \cdot \mathbf{v}) \text{ for all } \mathbf{v} \in \mathbf{V}_h.$$

(discrete) Helmholtz decomposition

$$\mathbf{V}_h = \mathbf{grad}_h S_h \oplus \mathbf{curl}\, W_h.$$

This decomposition is orthogonal with respect to both the L_2 and $H(\text{div})$ inner products. Using the orthogonality, one can easily show that the two summand spaces $\mathbf{grad}_h S_h$ and $\mathbf{curl}\, W_h$ are invariant under A and A_δ, where

$$\begin{aligned} (A\boldsymbol{\sigma}, \boldsymbol{\tau}) &= (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + (\boldsymbol{\sigma}, \boldsymbol{\tau}), \\ (A_\delta \boldsymbol{\sigma}, \boldsymbol{\tau}) &= (\nabla \cdot \boldsymbol{\sigma}, \nabla \cdot \boldsymbol{\tau}) + \delta(\boldsymbol{\sigma}, \boldsymbol{\tau}). \end{aligned}$$

Note that

$$A_\delta = A + (\delta - 1)I.$$

Iterative solver (submitted, Ku and Reichel)

$$(A + (\delta - 1))\boldsymbol{\sigma}_h = A_\delta \boldsymbol{\sigma}_h = -\mathbf{grad}_h(f + \delta u_H^G),$$

i.e.

$$\boldsymbol{\sigma}_h = (1 - \delta)A^{-1}\boldsymbol{\sigma}_h - A^{-1}\mathbf{grad}_h(f + \delta u_H^G).$$

Hence, we define iterative method as

$$\boldsymbol{\sigma}_{n+1} = (1 - \delta)A^{-1}\boldsymbol{\sigma}_n - A^{-1}\mathbf{grad}_h(f + \delta u_H^G).$$

Note that $\boldsymbol{\sigma}_h \in \mathbf{grad}_h S_h$ and $\|A^{-1}\| \leq 1$. Moreover, $\|A^{-1}|_{\mathbf{grad}_h S_h}\| << 1$.

Performance for different δ values with $h = \frac{1}{128}$.

δ	h^2	h^4	h^6	h^8	h^{10}
$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^D\|$	**0.0012**	**0.0012**	**4.0891**	**4.3579**	**4.3579**
$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^I\|$	**0.0012**	**0.0012**	**0.0012**	**0.0012**	**0.0012**
# of iterations	**8**	**8**	**8**	**8**	**8**

The New reduced method provides a good alternative for efficient and accurate approximation of the flux variables .

Thank you