A Simplified Stoichiometric Model of Nutrient-Mediated Pathogen Dynamics

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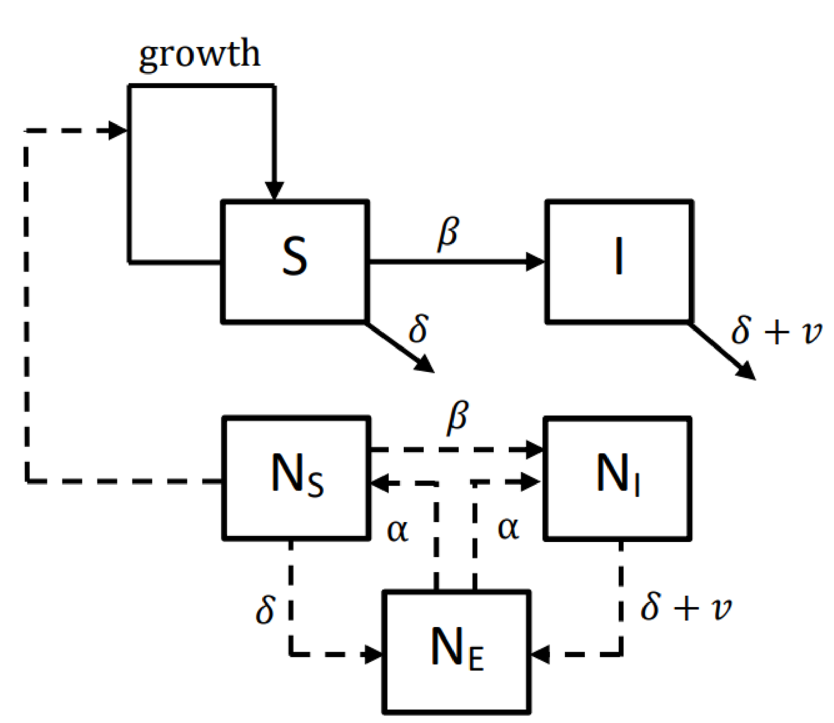
Overview

How do the spread of a pathogen and the ratio of nutrients in an environment affect each other and the population dynamics of a primary producer? The relationship between the spread of a pathogen and the ratio of nutrients in an environment can reveal important feedback loops within a population that may otherwise be overlooked. To better understand the relationship among disease dynamics, nutrient cycling, and population dynamics within an ecosystem, we use a stoichiometric model consisting of five equations. Due to the difficulty of analyzing five equations mathematically, our goal was to simplify the model to two equations. We find that assuming an instantaneous uptake of nutrients from the environment caused the system to lose its limit cycles, so the buffer of environmental nutrients in the system must be essential to the long term cycle behavior.

Model Description

In this model, S represents the carbon mass of the susceptible population, I represents the carbon mass of the infected population, N_S is the nitrogen mass of the susceptible population, N_I is the nitrogen mass of the infected, and N_E is the free nitrogen mass in the system.



Parameter	Description	Value
α	Nutrient uptake rate	.2273 umol N/L (varies)
N_T	Total nutrient in the system	100 umol N/L
μ	Susceptible maximal growth rate	1.35 /day
δ	Natural death rate	0.01 /day
v	Disease-induce death rate	0.09 /day
q	Susceptible minimal N:C	0.0303 umol N/umol C
K	Susceptible carrying capacity limited by light	200–1000 umol C/L
β	Transmission rate	0.0492 L/umol C/day

Full Mathematical Model

$$\frac{dS}{dt} = \underbrace{\mu\left(1-\frac{S+I}{min\{K\frac{\mu}{\mu-\delta},\frac{N_S}{qS}(S+I)\}}\right)S}_{\text{Growth}} - \underbrace{\beta SI}_{\text{Infection}} - \underbrace{\delta S}_{\text{Natural Death}} \quad (1a)$$

$$\frac{dI}{dt} = \underbrace{\beta SI}_{\text{Infection}} - \underbrace{(\delta+v)I}_{\text{Disease Death}} \quad (1b)$$

$$\frac{dN_S}{dt} = \underbrace{\alpha N_E S}_{\text{Linear Uptake}} - \underbrace{\beta I N_S}_{\text{Infection}} - \underbrace{\delta N_S}_{\text{Natural Death}} \quad (1c)$$

$$\frac{dN_I}{dt} = \underbrace{\alpha N_E I}_{\text{Linear Uptake}} + \underbrace{\beta I N_S}_{\text{Infection}} - \underbrace{(\delta+v)N_I}_{\text{Disease Death}} \quad (1d)$$

$$\frac{dN_E}{dt} = \underbrace{-\alpha N_E(S+I)}_{\text{Linear Uptake}} + \underbrace{\delta N_S}_{\text{Natural Death}} + \underbrace{(\delta+v)N_I}_{\text{Disease Death}} \quad (1e)$$

We note that $N_T = N_S + N_I + N_E$ and $N_S' + N_I' + N_E' = 0$, so it is implicit that the total nutrients are constant in the system. This renders the N_E equation redundant and simplifies the system.

Fast Nutrient Dynamics

To simplify the model, we assume that the nutrient flow rate is on a much faster timescale than the rate of change of the population, so we assume that $\alpha \to \infty$, and that N_S and N_I can be approximated by equilibria in terms of S and I by the quasi steady state argument.

To find these equilibria, we set $N_S' = 0$ and $N_I' = 0$, and solve for values of N_S and N_I as functions of S and I:

$$N_S = \frac{\alpha N_T S(\delta+v)}{\delta^2+\beta I v+\delta(\beta I+\alpha(I+S)+v)+\alpha(\beta I(I+S)+Sv)}$$

$$N_I = \frac{\alpha N_T I(\delta+\beta(I+S))}{\delta^2+\beta I v+\delta(\beta I+\alpha(I+S)+v)+\alpha(\beta I(I+S)+Sv)}.$$

Because nutrient dynamics are much faster than the population timescale, we can approximate uptake from the environment to be immediate, i.e. $\alpha \to \infty$. Thus, we let

$$N_S \to \frac{N_T S(\delta+v)}{\delta(I+S)+\beta I(I+S)+Sv} \text{ and } N_I \to \frac{N_T I(\delta+\beta(I+S))}{\delta(I+S)+\beta I(I+S)+Sv}. \quad (2)$$

Note that as $\alpha \to \infty$, $N_S + N_I \to N_T$, so $N_E \to 0$. Therefore, we define N_S and N_I as in (2) to reduce the system to two equations:

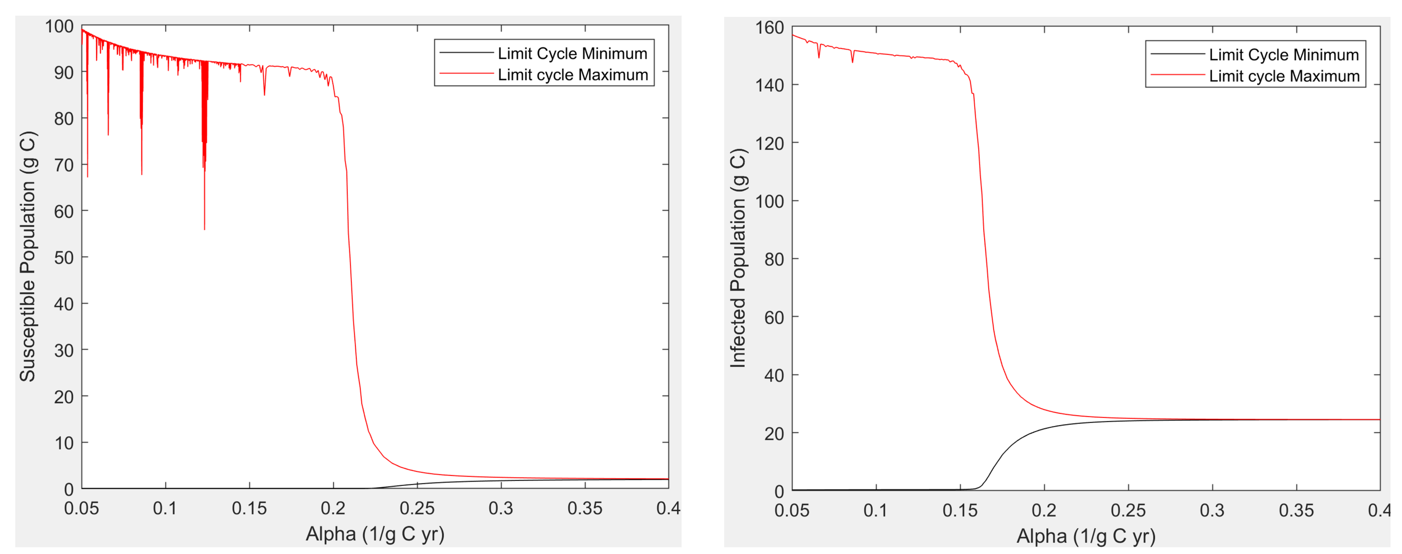
$$S' = \mu\left(1-\frac{S+I}{min\{K\frac{\mu}{\mu-\delta},\frac{N_T(\delta+v)}{q((\delta+\beta I)(I+S)+Sv)}(S+I)\}}\right)S - \beta SI - \delta S \quad (3a)$$

(3b)

$$I' = \beta SI - (\delta+v)I. \quad (3c)$$

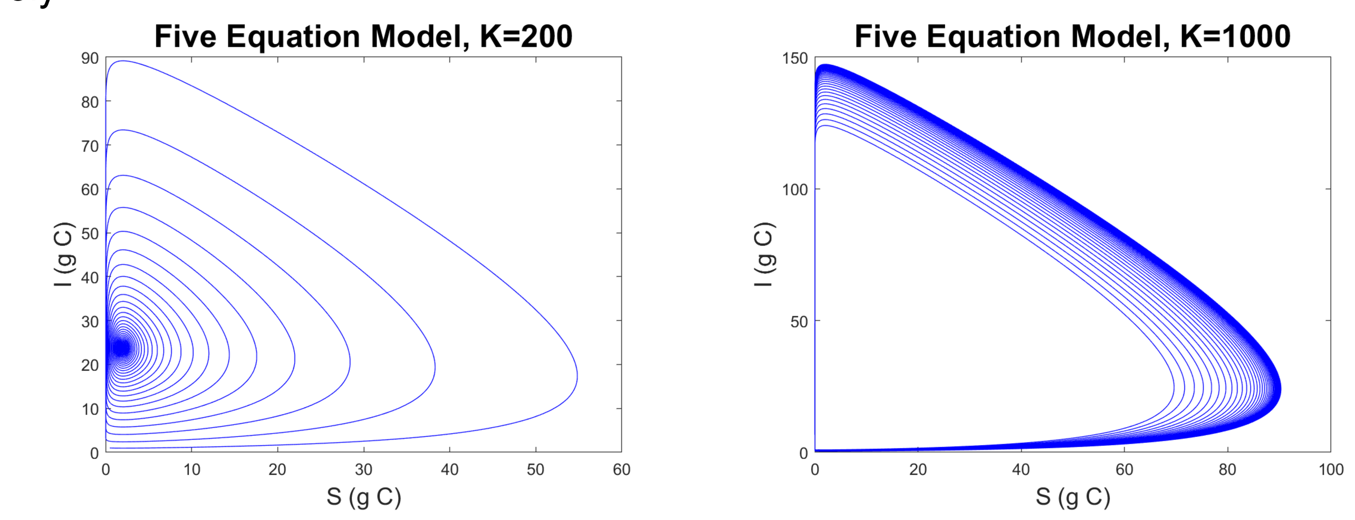
Bifurcation Diagrams

Bifurcation diagrams with respect to α of the susceptible population and the infected population

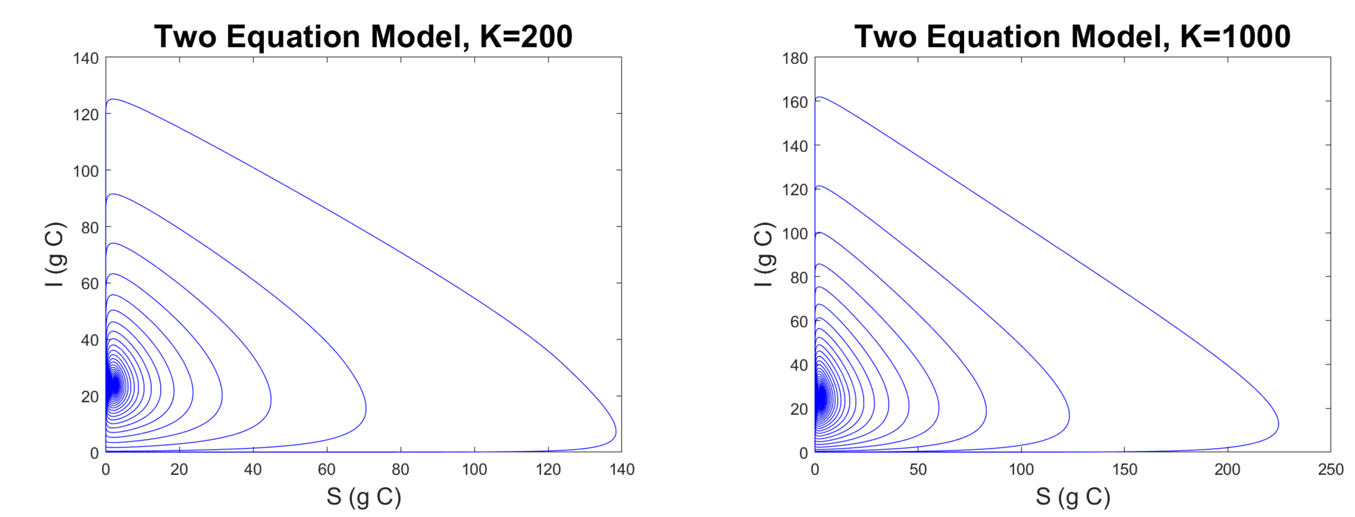


Phase Plane Simulations

Simulations of the five equation system (1) with light-limited growth and nutrient-limited growth, respectively



Simulations with the same parameters, but instead using the two equation model (3)



Equilibrium Stability Analysis

Both of the models have two boundary equilibria and one internal equilibrium. Using Jacobian matrices, we found that if the growth of the system is light-limited, in either model the internal equilibrium will be a sink. However, if the growth is nutrient-limited, the internal equilibrium will be a source in the five equation model and a sink in the two equation model.