A Nonlinear Einstein Paradigm for Brownian Motion in Chemotactic model exhibiting traveling band in abundant of chemical substrate

Rahnuma Islam

Graduate student; Texas Tech University

Introduction

"Chemotaxis" is a biological phenomena by which organisms change their state of movements either toward or away from the chemical substance. In 1960s, Adler and his associates put bacteria in one end of a capillary tube and energy source such as galactose, glucose, threonine or serine with oxygen in the other end, then a band of bacteria was observed travelling in the tube.

We consider the classic Einstein's Brownian motion paradigm to deduce a thought experiment of movement of living organism which has been shown to exhibit travel band. Chemotactic response of the bacteria and the random motion of bacteria arising from chemotactic response has been taken into account in the presence of unlimited chemical substrate consumed by the organism. In addition, we also consider the formation of bacterial crowd via interactions within or between the bacterial community.

Einstein's random paradigm

x : position of particles
t : time
τ : time interval between the collision of two particles
Δ : distance traveled during the time interval τ
φ : probability density function of particle jumps
Then, the number of particle found at time $t+\tau$ between two planes perpendicular to the x-axis, with abscissas x and $x+\Delta x$:

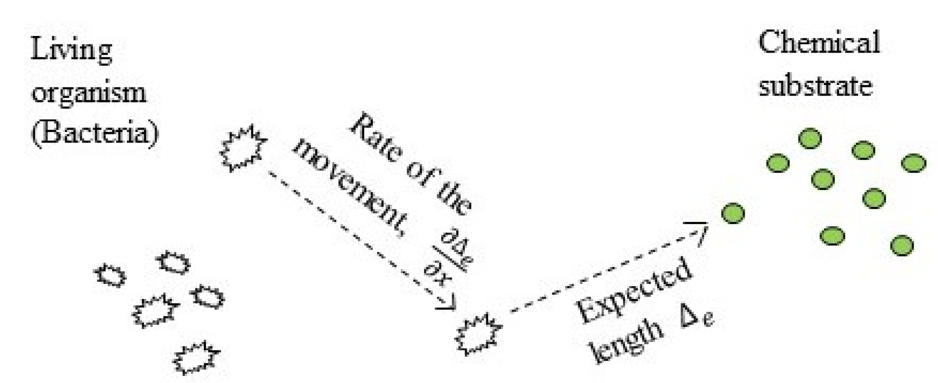
$$u(x,t+\tau)\cdot dx = \left(\int_{-\infty}^{\infty} u(x+\Delta,t)\varphi(\Delta)d\Delta + \int_t^{t+\tau} f(x,\xi)d\xi\right)\cdot dx \tag{1}$$

Using Caratheodory theorem, we get

$$\tau\frac{\partial u}{\partial t} - \Delta_e\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x}\int_{-\infty}^{\infty}\Delta\varphi(\Delta)d\Delta + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\int_{-\infty}^{\infty}(\Delta-\Delta_e)^2\varphi(\Delta)d\Delta + \int_t^{t+\tau} f(x,\xi)d\xi \tag{2}$$

Derivation of chemotaxis model

Let $u(x,t)$ and $v(x,t)$ are the density of bacteria and critical substrate per unit volume.



Hypotheses of bacteria:

1. Chemotactic response of the bacteria:
$$\int_{-\infty}^{\infty}\Delta\varphi(\Delta)d\Delta = \Delta_{e,u}(v) = -\beta\frac{\partial \ln v}{\partial x}$$
β :a positive chemotactic coefficient
2. Random motion of bacteria:
$$\int_{-\infty}^{\infty}(\Delta-\Delta_e)^2\varphi(\Delta)d\Delta = \sigma_u^2(v) = \mu$$
μ : the motility parameter or diffusion coefficient of the bacteria
3. Bacterial interactions:
$$\int_t^{t_u+\tau} f_u(x,\xi)d\xi = \tau u\gamma_0\frac{\partial\Delta_{e,u}(v)}{\partial x} = -u\tau\gamma\frac{\partial^2 \ln v}{\partial x^2}$$
γ_0 : the minimum amount of rate of autoinducers that should be present for bacteria to secret signal molecule

Hypotheses of chemical substrate:

1. No Chemical bonding:
$$\Delta_{e,v} = 0,$$
2. No random motion of substrate:
$$\sigma_v^2 = 0,$$
3. Consumption of substrate by bacteria:
$$\int_t^{t_v+\tau} f_s(u,v)dv = \tau F_v(u,v) = -\tau k u,$$
k : the rate of consumption of the substrate.

Chemotactic model for unlimited substrate

$$L_0 u = \tau\frac{\partial u}{\partial t} + 2\beta\frac{\partial \ln v}{\partial x}\frac{\partial u}{\partial x} - \frac{\mu}{2}\frac{\partial^2 u}{\partial x^2} + \tau\gamma u\frac{\partial^2 \ln v}{\partial x^2} \tag{3}$$

$$L_0 v = \frac{\partial v}{\partial t} + ku \tag{4}$$

Model when $\gamma = \frac{2\beta}{\tau}$

Upon setting $\gamma = \frac{2\beta}{\tau}$, our chemotactic model gives the classic Keller-Segel model and the analytic solutions are

$$u_0 = \frac{2c^2\tau k^{-1}}{4\beta-\mu}v_\infty\left(e^{-\frac{2\tau c}{\mu}\zeta}+1\right)^{-\frac{d}{d-1}}e^{-\frac{2\tau c}{\mu}\zeta} \tag{5}$$

$$v_0 = \frac{v_\infty}{\left(1+e^{-\frac{2\tau c}{\mu}\zeta}\right)^{\frac{1}{d-1}}} \tag{6}$$

with $d = \frac{4\beta}{\mu} > 1$ and traveling speed

$$c = \frac{k\int_{-\infty}^{\infty}u(\zeta)d\zeta}{v_\infty} \tag{7}$$

Model when $\gamma \neq \frac{2\beta}{\tau}$

Let

$$\max\left|\frac{\partial^2 \ln v_0}{\partial x^2}\right| = B = \frac{1}{d-1}\frac{\tau^2 c^2}{\mu^2} > 0 \tag{8}$$

a constant. Then there exists constants $\lambda_\pm$ such that

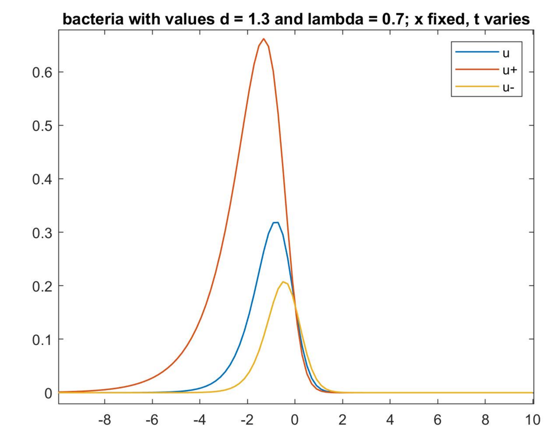
$$u_\pm(x,t) = e^{\lambda_\pm t}u_0(x,t) \text{ and } v_\pm = e^{\lambda_\pm t}v_0(x,t) \tag{9}$$

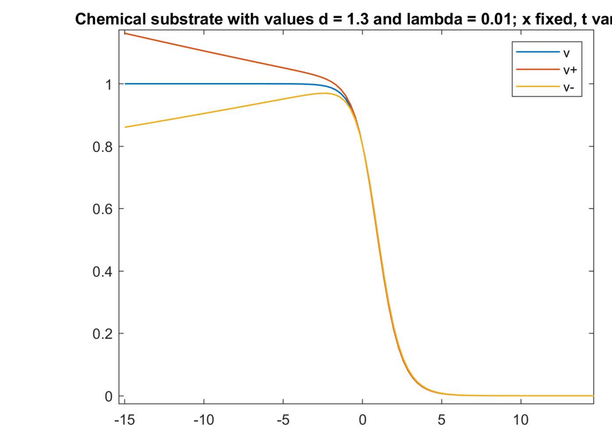
solves the partial differential inequality

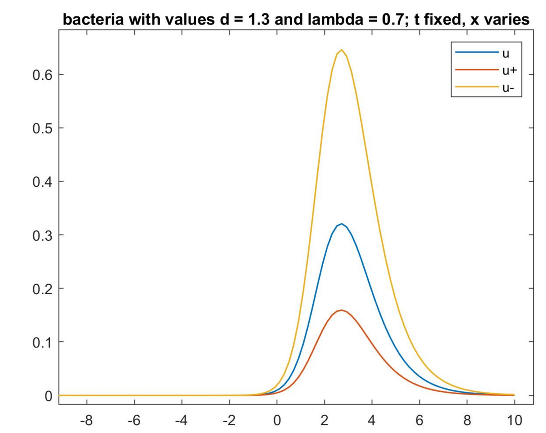
$$L_\alpha u_+ \le 0 \text{ and } L_\alpha u_- \ge 0$$
$$L_\alpha v_+ \le 0 \text{ and } L_\alpha v_- \ge 0$$

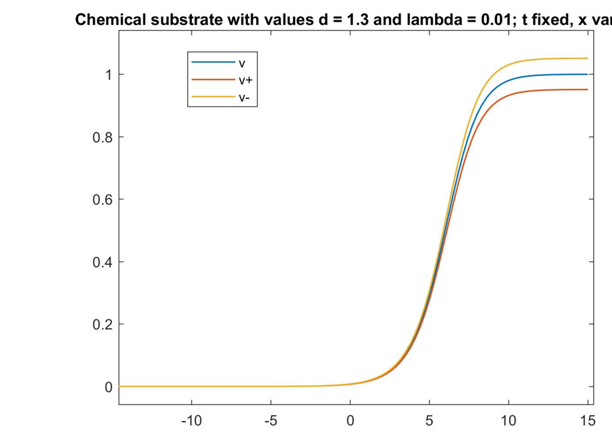
As a result, $u_\pm$ and $v_\pm$ are the upper and lower estimates of the solution u and v respectively.

Numerical analysis









Conclusion

We derive the chemotactic model motivated by Einstein's random walk model exhibiting traveling band in environment with unlimited supply of food. If $2\beta = \gamma\tau$, then our model derive the Keller-Segel model with analytical solution. But, there is no guarantee that the threshold value to initiate chemotactic response to bacteria and amount of inducers to create bacterial communication will present in the same amount in any environment. In the later case, analytical solution is not achievable. But we obtain estimates on the solution from above and below proving that the traveling band solution of the first model is sandwiched between these two estimates.

References

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