Einstein Brownian Motion with singularity and Fractional Laplacian on the boundary

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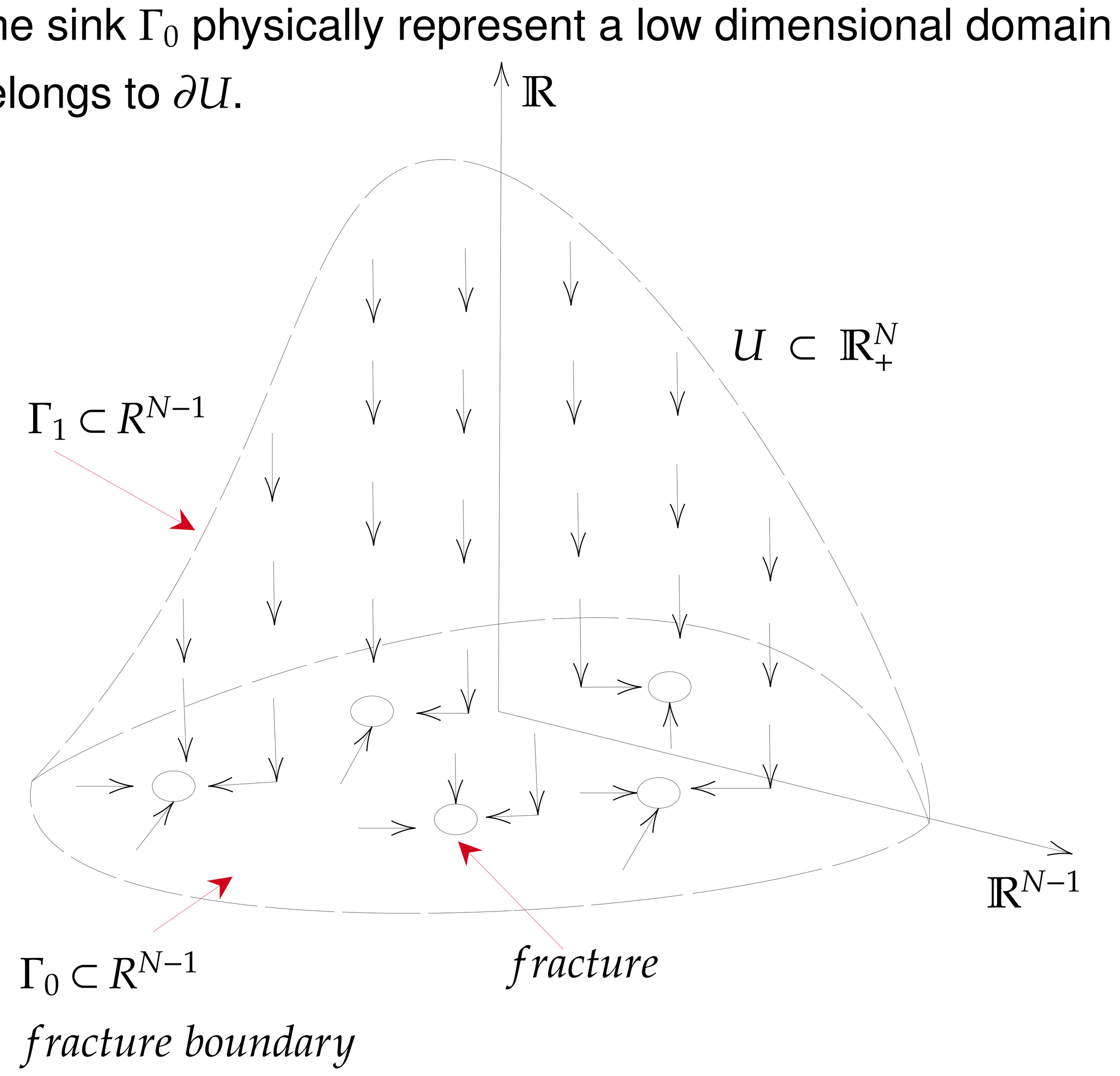
1. Introduction

We present recent result on the modeling of the flow in porous media using Einstein's thought experiment of the jumps of the particles. The key parameter of our problem is "super fast" flow towards of the fracture-boundary of the media U. To utilize the Einstein concept we model fluid as a family of particles which are transporting due to diffusion and drift towards suction $\Gamma \subset \bar{U}$. Let number of particle per unit volume at point of observation x at time t to be equal $u(x,t)$. Dynamics of the process is such that flow toward designated suction is not comparably higher than random jumps of the particles. According to the Einstein's paradigm the length - Δ of the jumps without collision is the events. In case of the Brownian motion Einstein assumed that that the expected value of free jumps Δ_e equals to 0, which is not valid for flows with a suction.

2. Mass Conservation

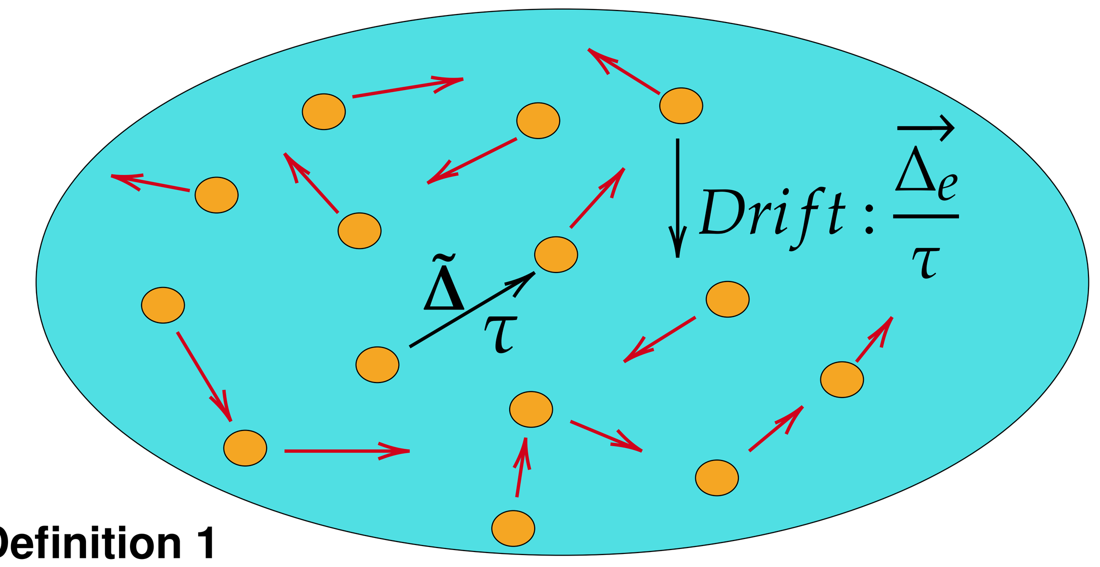
We consider $N \geq 2$ dimensional flow in the medium $U \subset \mathbb{R}^N_+ := \{x \in \mathbb{R}^{N-1}, x_N > 0\}$ of the fluid towards sink $\Gamma_0 \subset \mathbb{R}^{N-1} \times \{x_N = 0\}$, which belongs to the boundary of the domain of the flow. The sink Γ_0 physically represent a low dimensional domain belongs to ∂U.



In the our thought experiment fluid modeled as collection of particles featured by density at points of the observation $x \in \mathbb{R}^N$ at time t. We will use Einstein paradigm to develop a deterministic mathematical model of the random transport of fluid's particles towards sink fracture

Assumption 1

1. *Denote $\mathbb{P}(\tau)$ to be the set of vectors with non-colliding jumps of the particles P corresponding to time interval τ. We call $\vec{\Delta} = (\Delta_1, \cdots, \Delta_N)^T$ to be a " vector of free jump of particles P" if $\vec{\Delta} \in \mathbb{P}(\tau)$.*
2. *Time interval of free jumps τ, expected vector $\vec{\Delta}_e$ of a free jump $\vec{\Delta}$ and probability density function of free jump $\varphi(\vec{\Delta})$ are the only parameters which characterise process of free jumps. Note that in a view of the definition of the set $\mathbb{P}(\tau)$, if $\vec{\Delta} \notin \mathbb{P}(\tau)$ then $\varphi(\vec{\Delta}) = 0$*



Definition 1

Expected vector of free jumps

$$\vec{\Delta}_e \triangleq (\Delta_e^1, \Delta_e^2, \ldots, \Delta_e^N)^T \quad where \quad \Delta_e^i \triangleq \int_{\mathbb{P}(\tau)} \Delta_i \varphi(\vec{\Delta}) d\vec{\Delta} \qquad (1)$$

Standard Co-variance matrix of a free jump

$$\sigma_{ij}^2 \triangleq \int_{\mathbb{P}(\tau)} \left(\Delta_i - \Delta_e^i\right)\left(\Delta_j - \Delta_e^j\right) \varphi(\vec{\Delta}) d\vec{\Delta} \qquad (2)$$

We postulate generalized Einsteins Axiom for the number of particles found at time $t+\tau$ in the control volume dv contained point x by :

Axiom of Mass conservation 1

$$u(x, t+\tau) \cdot dv = \left(\int_{\mathbb{P}(\tau)} u(x+\vec{\Delta}, t) \varphi(\vec{\Delta}) d\vec{\Delta}\right) \cdot dv \qquad (3)$$

as it says that at any given point in space x at time $t+\tau$ we will observe all particles with free jumps from the same x.

3. Singular IBVP

Let ζ be a multi-index. Assume that $u(x,t) \in C^{2,1}_{x,t}$. We apply Taylor's Expansion, and using (1) and (2) we get

$$u(x,t+\tau) - u(x+\vec{\Delta}_e, t) = \sum_{i,j=1}^{N} a_{ij}(x,t) u_{x_i x_j}(x+\vec{\Delta}_e, t) + R_\zeta \qquad (4)$$

where R_ζ is the remainder term in Taylor's Expansion. Let

$$a_{ij}(x,t) = \sigma_i^2(x,t)/2 \text{ if } i=j \quad \text{and} \quad a_{ij}(x,t) = \sigma_{ij}^2(x,t) \text{ o/w} \qquad (5)$$

LHS and RHS in the above equation (4) are defined in different points. To fix this ambiguity let assume that $u(x,t) \in C^{3,1}_{x,t}$ in Eq (4). Then by Carathéodory's criterion $\exists$ function $\vec{\psi}_{ij}^{xxx} : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ such that

$$\sum_{i,j=1}^{N} a_{ij}(x,t) u_{x_i x_j}(x+\vec{\Delta}_e, t) = \sum_{i,j=1}^{N} [\vec{\psi}_{ij}^{xxx}(x, \vec{\Delta}_e, t) \cdot \vec{\Delta}_e] a_{ij}(x,t) + \sum_{i,j=1}^{N} a_{ij}(x,t) u_{x_i x_j}(x,t). \quad (6)$$

Again by Carathéodory's criterion for $u(x,t) \in C^{2,2}_{x,t}$ in Eq (4) $\exists$ functions $\psi^t, \psi^{tt}, \psi_i^x \in \mathbb{R}$ and $\vec{\psi}_i^{xx} \in \mathbb{R}^N$ such that

$$u(x,t+\tau) - u(x+\vec{\Delta}_e, t) \approx -\sum_{i=1}^{N} [\vec{\psi}_i^{xx}(x, \vec{\Delta}_e, t) \cdot \vec{\Delta}_e] \Delta_e^i + \tau \psi^t(x,t,0) - \sum_{i=1}^{N} \psi_i^x(x,0,t) \Delta_e^i \quad (7)$$

Here ψ^t and ψ_i^x are such that $\lim_{\nu \to 0} \psi^t(x,t,\nu) = u_t(x,t)$ and $\lim_{|s| \to 0} \psi_i^x(x,s,t) = u_{x_i}(x,t)$ respectively. Using (7) in LHS of (4), and (6) in RHS of (4), and approximating the first order terms, equation (4) becomes

$$u_t - \frac{1}{\tau}\sum_{i=1}^{N} u_{x_i} \Delta_e^i - \frac{1}{\tau}\sum_{i,j=1}^{N} a_{ij} u_{x_i x_j} \approx \frac{1}{\tau} \underbrace{\left[\sum_{i=1}^{N} [\vec{\psi}_i^{xx}(x, \vec{\Delta}_e, t) \cdot \vec{\Delta}_e] \Delta_e^i + \sum_{i,j=1}^{N} [\vec{\psi}_{ij}^{xxx}(x, \vec{\Delta}_e, t) \cdot \vec{\Delta}_e] a_{ij}(x,t) + R_\zeta\right]}_{B}$$

Assumption 2 *u and $\vec{\Delta}_e$ be such that $B(x,t,\vec{\Delta}_e) = 0$ at each point $x \in U \in \mathbb{R}^N_+$ at time $t \in \mathbb{R}$.*

Axiom 1 *Let the expected value of free jumps is such that variance of length of free jump incomparably smaller than expected value of free jump in the direction x_N assuming that*

$$\Delta_e^i = 0,\ i = 1, \cdots, N-1, \quad \Delta_e^N = \frac{\beta}{x_N^\alpha}, \quad \text{and} \quad \sigma_{i,j} = k_2 \delta_{ij} \qquad (8)$$

where $k_2 > 0$, α , $\beta \geq 0$ and δ_{ij} is the Kronecker symbol. Then (8) gives

$$\tau u_t - \left(\frac{\beta}{x_N^\alpha}\right) u_{x_N} - \sum_{i,j=1}^{N} a_{ij}(x,t) u_{x_i x_j} = 0 \qquad (9)$$

In this work we assume that the process of the "free' jumps is characterised only by the time interval of free jump (τ).

Axiom 2 *Let $\gamma > 0$. We call (9) as a dynamic process of γ jumps if*

$$\tau \approx \frac{1}{u^\gamma} ;\ \gamma > 0. \qquad (10)$$

This constrain intuitively justify when the process has few number of particles. By Axiom 3, Axiom 1, and Assumption 2 follows that u satisfies IBVP

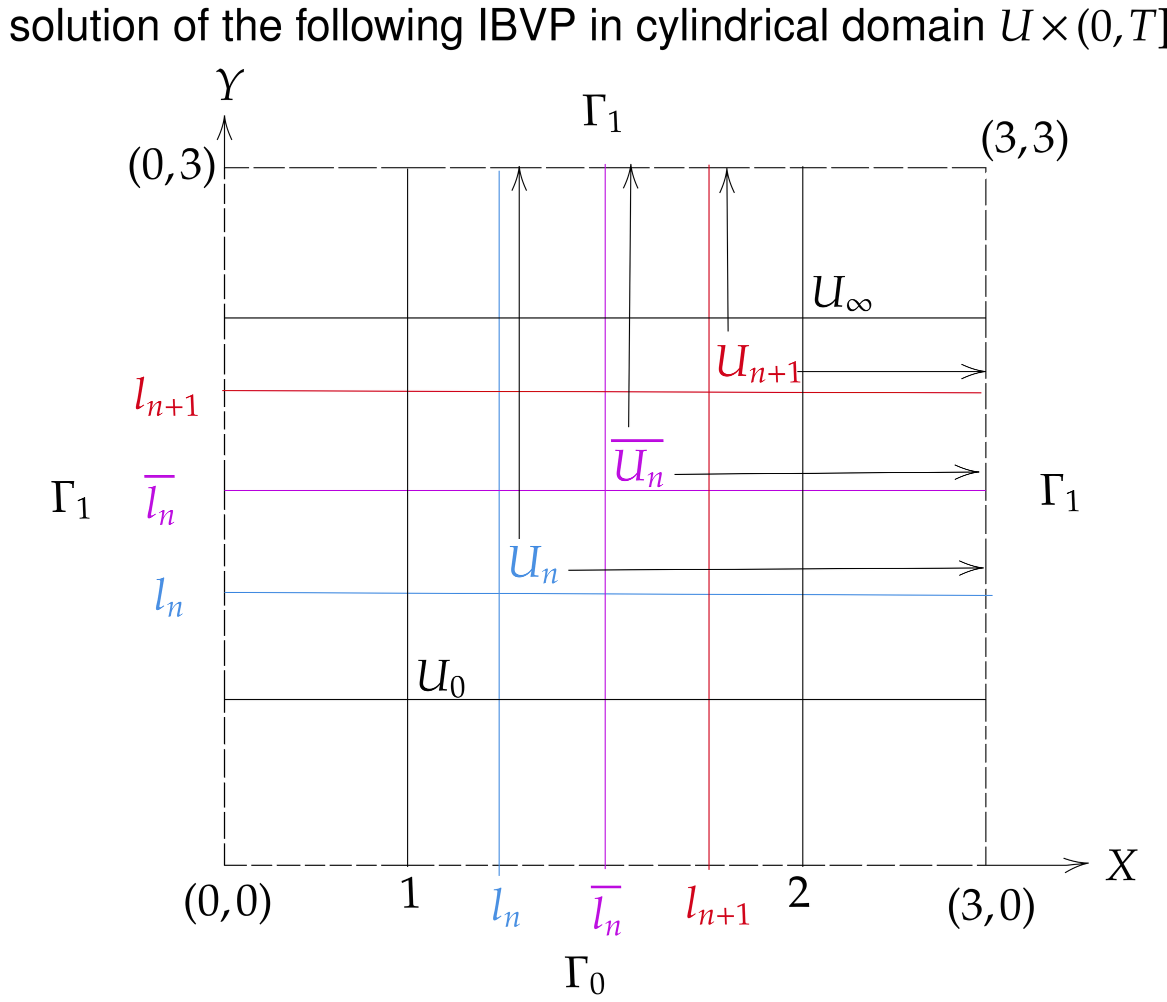
$$u_t - \left(\frac{\beta}{x_N^\alpha}\right) u^\gamma u_{x_N} - k_2 u^\gamma \Delta u = 0 \text{ in } U \times (0,T] \qquad (11)$$

$$u(x,0) = u_0(x) \text{ in } U \qquad (12)$$

$$u(x,t) = 0 \text{ on } \{\partial U \setminus \Gamma_0\} \times (0,T]. \qquad (13)$$

4. Localization property

Let $U \subset \mathbb{R}^2$ be a rectangular open bounded domain with $0 < x < 3$ and $0 < y < 3$. Define $0 \leq u(X,t) \leq M$ be a classical solution of the following IBVP in cylindrical domain $U \times (0,T]$:



Let $u_0 \neq 0$ in $0 < x < \frac{1}{2}$ & $0 < y < \frac{1}{2}$. Consider the sequences

$$l_n = 2 - \frac{1}{2^n},\ \overline{l_n} = \frac{l_n + l_{n+1}}{2} ;\ n = 0, 1, 2, \ldots. \qquad (14)$$

Define following iterative sets

$$U_n \triangleq \{X \mid l_n \leq x < 3,\ l_n \leq y < 3\} \qquad (15)$$

$$\overline{U_n} \triangleq \{X \mid \overline{l_n} \leq x < 3,\ \overline{l_n} \leq y < 3\} \qquad (16)$$

Note that $U_\infty \subset \cdots U_{n+1} \subset \overline{U_n} \subset U_n \cdots \subset U_0$. Define cut-off function

$$\eta_n = 0 \text{ in } U \setminus U_n \text{ and } \eta_n = 1 \text{ in } \overline{U_n}.\ |\nabla \eta_n| \leq c 2^n. \qquad (17)$$

Lemma 1 *Let u solves the IBVP in $\mathbb{R}^2$. Let θ and p be fixed such that $\gamma + \theta > p > 2$. Then*

$$\sup_{0<t<T} \int_{U_{n+1}} u^{\theta+1} dX + k_2(\theta+1)(\gamma+\theta-p) \int_0^T \int_{U_{n+1}} |\nabla u|^2 u^{\theta+\gamma-1} dX dt \leq 4^n (\theta+1) M_c M_\alpha \int_0^T \int_{U_n} u^{\theta+\gamma+1} dX dt \ (18)$$

Definition 2

$$\text{Let } v \triangleq u^{(\theta+\gamma+1)/2} \quad \text{with} \quad \lambda \triangleq 2(\theta+1)/(\theta+\gamma+1) \qquad (19)$$

For any $n = 0, 1, 2\ldots$ we define

$$I_n(T) \triangleq \sup_{0<\tau<T} \int_{U_{n+1}} v^\lambda dX + \int_0^T \int_{U_{n+1}} |\nabla v|^2 dX dt \qquad (20)$$

Lemma 2 *Assume all conditions in Lemma 1 are satisfied and u and v be as in (19). Then $\exists$ $C_L, \epsilon_0 > 0$ and $b_L > 1$ s.t for any $n = 1, 2, \ldots$*

$$I_n(T) \leq t^{1-\zeta} C_L b_L^{n-1} I_{n-1}^{1+\epsilon_0}(T) \quad ;\ 0 < \zeta < 1. \qquad (21)$$

where $\epsilon_0 = 4\gamma/\gamma + 4(\theta+1)$ and $b_L = 4$. Next we prove localization property using Ladyzhenskaya iterative Lemma.

5. Main Theorem

Theorem 1 *Assume that all conditions in Lemma 2 satisfy. If*

$$I_0(T) \leq 2^{-\left(\frac{2}{\epsilon_0^2}\right)} \left(\frac{C_L}{G}\right)^{-\frac{1}{\epsilon_0}} T^{-\frac{1}{\epsilon_0}} \qquad (22)$$

Then $I_n(T) \to 0$ as $n \to \infty$.

Assume that $u(x,t) \geq 0$ be a classical solution of IBVP (11) define on bounded $U \in \mathbb{R}^2$ with $\mathrm{supp}\, u_0(x)$. Theorem 1 provides a constrain in form of integral inequality (22) on the class on solution which guaranties finite speed of propagation : $u(x,t) = 0$ for all $0 \leq t \leq T$, and $x \in U \setminus \mathrm{supp}(u_0)$. Let u and v relate as in (19) satisfies condition (22) then $v(x,t) = 0$ a.e. in $U_\infty = \bigcap_{i=n}^{\infty} U_i$ for any $t \leq T$. Consequently

$$u(x,t) = 0 \quad \text{a.e. in} \quad U \setminus \mathrm{supp}\, u_0 \quad \text{for any} \quad t \leq T \qquad (23)$$

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