**Introduction**

An asymptomatic case is an individual who tests positive but experiences no symptoms throughout the course of infection. Asymptomatic infection is very common for many infectious diseases including COVID-19, Ebola, influenza, cholera, Zika, fever, and dengue fever. For example, a meta-analysis based on 28 studies estimated that 25% COVID-19 infections are asymptomatic. Asymptomatic infections are hard to detect but may transmit the disease to others, acting as silent spreaders.

Global travel and tourism accelerate the spread of infectious diseases and constitute a major challenge for disease prevention and control (e.g., 2009 H1N1 pandemic, 2014-16 Ebola outbreak, 2015-16 Zika virus epidemic, 2019-20 COVID-19 pandemic). Entry and exit restrictions can hardly detect asymptomatic travelers who are more likely to spread the infectious agent from one area to another due to their uninterrupted mobility. There are many epidemic models on asymptomatic infection (e.g., Kemper, 1978; Arino, 2008) or human movement (e.g., Revachev and Longini, 1985; Lewis et al., 2006; Allen et al., 2008), but few considered their joint effect.

**Theorem 1.** For model (1), if $R_0 \leq 1$, then the disease-free equilibrium $E_0$ is globally asymptotically stable; if $R_0 > 1$, then the disease is uniformly persistent and there exists at least one endemic equilibrium $E^* = (\bar{S}, \bar{I}, \bar{A}, \bar{R})$.

**II. Dependence of $R_0$ on Dispersal Rates.** In case of no asymptomatic infection, model (1) is reduced to an SIR patch model and its reproduction number is the same as that of an SI patch model (e.g., Allen et al., 2007; Gao and Ruan, 2011).

**Theorem 2.** (Gao, 2019; Gao and Dong, 2020) For model (1) with $\beta_i = 1$ for all $i \in \Omega$, the basic reproduction number $R_0(d_i) = \rho(F_1(i))$ and the spectral bound $\lambda_1(d_i)$ are strictly decreasing and strictly convex in $d_i \in [0, \infty]$ if $\beta_i = \beta_i(\gamma_i) = \gamma_i(d_i)$ and $\beta_i = \gamma_i$ are respectively nonconstant in $i \in \Omega$ and constant otherwise.

However, for the SIAR patch model (1) with $n = 2$, it is shown that $R_0$ is either strictly decreasing or strictly increasing or constant with respect to $d_1$ and $d_2$.

Nonmonotone dependence of $R_0$ on dispersal rates can occur with more patches.

**Theorem 3.** Suppose $\beta_i = \beta_1$ and $\gamma_i = \gamma_1$ for all $i \in \Omega$, and $I^*_i$ and $A^*_i$ are symmetric. Then the basic reproduction number $R_0(d_i)$ of model (1) is constant in terms of $d_i$ if $(F_1(i) - d_i I^*_i A^*_i)^2 1 = R_0(0) 1_i$ and strictly decreasing otherwise.

**Proposition 4.** Assume that (i) $\beta_1 = \beta_1$ and $\gamma_1 = \gamma_1$ for all $i \in \Omega$, (ii) $\beta_1 = \beta_1$ and $\gamma_1 = \gamma_1$ for all $i \in \Omega$, (iii) there is a positive diagonal matrix $C$ such that $C\gamma_i$ is symmetric. Let $\alpha_i$ be a positive eigenvector of $C$ corresponding to eigenvalue zero. Then the basic reproduction number $R_0(d_i)$ of model (1) is constant in terms of $d_i$ if $(F_1(i) - d_i I^*_i A^*_i)^2 1 = R_0(0) 1_i$ and strictly decreasing otherwise.

**Theorem 4.** For model (1), if $d_1 = 0$ (or $d_2 = 0$), then the basic reproduction number $R_0$ is strictly decreasing with respect to $d_2$ (or $d_1$) whenever $\alpha_1 > 0$ is constant in $i \in \Omega$ and constant otherwise.

**III. Dependence of $R_0$ on Dispersal Rates.** We consider under what conditions $R_0(d_i)$ and $R_0(d_i, d_j)$ are constant, respectively.

**Proposition 3.** For model (1), the following statements on $R_0$ hold.

- $R_0$ is independent of dispersal if both $\beta_i^0$ and $\gamma_i^0$ are constant in $i \in \Omega$.

- $R_0$ is independent of dispersal if $I^*_i$ is constant in $i \in \Omega$ and $\sum (F_1(i) - d_i I^*_i A^*_i)^2 1 = R_0(0) 1_i$ for any $d_i, d_j \geq 0$.

- $R_0$ is independent of dispersal if $\gamma_1$ is constant in $i \in \Omega$ and $\gamma_1 = \gamma_1$ for some $k > 0$. where $\gamma_1$ is a right positive eigenvector with eigenvalue zero of matrix $\alpha_i^0 \bar{I} - \bar{A}_1$ but not conversely.

Discussion

We propose an SIAR patch model to address asymptomatic infection and spatial heterogeneity. The reproduction number $R_0$ is defined and estimated. The relation between $R_0$ and dispersal rates is analyzed. Ignoring asymptomatic infections will underestimate the infection risk, and the level of underestimation significantly varies with dispersal rates. The inconsistency between $R_0$ and the nonsusceptible ratio in response to fast dispersal highlights the necessity of assessing the effectiveness of control measures with other quantities besides the basic reproduction number.

Numerical Simulations

**Example 1** Infection risk versus dispersal rates.

**Example 2** Ignoring asymptomatic infection underestimates the infection risk.